Numerical Differentiation and Integration

UNIT3

Yuba Raj Devkota

Unit 3 Numerical Differentiation and Integration **5 Hrs.**

Introduction to Numerical Differentiation, Newton's Differentiation Formulas, Numerical Integration (Trapezoidal Rule, Simpson's 1/3 rule, 3/8 rule); Romberg Integration; Numerical Double Integration.

Numerical Differentiation

The method of obtaining the derivative of a function using a numerical technique is known as numerical differentiation. There are essentially two situations where numerical differentiation is required. They are:

1. The function values are known but the function is unknown. Such functions are called tabulated functions.
2. The function to be differentiated is complicated and, therefore, it is difficult to differentiate.

Differentiating continuous function

$f'(x) =?$ *of a function* $f(x)$, *when the function itself is available.*

➢ ***Forward Difference Quotient***

We have, Taylor series is;

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots \quad \text{(i)}$$

Neglecting higher order of derivative, we get,

$$f(x+h) = f(x) + hf'(x)$$

$$\boxed{\therefore f'(x) = \frac{f(x+h)-f(x)}{h}}$$

Which is the first order *forward difference quotient*.

➢ ***Backward Difference Quotient***

We have, Taylor series is;

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots \quad \text{(i)}$$

Neglecting higher order of derivative, we get,

$$f(x-h) = f(x) - hf'(x)$$

$$\boxed{\therefore f'(x) = \frac{f(x)-f(x-h)}{h}}$$

Which is the first order *backward difference quotient*.

➢ ***Central Difference Quotient***

We have, Taylor series is;

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots \qquad \text{(i)}$$

Similarly,

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots \qquad \text{(ii)}$$

Subtracting eq. (ii) from eq. (i) and neglecting higher order derivative we get.

$$\boxed{f'(x) = \frac{f(x+h) - f(x-h)}{2h}}$$

Which is the *central difference quotient*.

Examples

1. Estimate approximate derivative of $f(x) = x^2$ at $x = 1$, for h=0.2 using the forward difference and central difference formula.

Sol^n:

Using the forward difference formula:

We have,

$$f'(x) = \frac{f(x+h)-f(x)}{h}$$

Therefore,

$$f'(1) = \frac{f(1+0.2)-f(1)}{0.2} = \frac{f(1.2)-f(1)}{0.2} = 2.2$$

Using the central difference formula:

We have,

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h}$$

Therefore,

$$f'(1) = \frac{f(1+0.2)-f(1-0.2)}{2\times0.2} = \frac{f(1.2)-f(0.8)}{0.4} = 2$$

2. Estimate the first derivative of f(x)=ln x at x=1 using the second order central difference formula.

<u>Sol</u>^{n}*:*

Given,

f(x)= ln x

Take h=0.1

We have,

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h}$$

$$\therefore f'(1) = \frac{f(1+0.1)-f(1-0.1)}{2*0.1} = \frac{f(1.1)-f(0.9)}{0.2} = = \frac{0.095-(-0.1054)}{0.2} = 1.002$$

Higher- order Derivatives

We have, Taylor series is;

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots \qquad \text{(i)}$$

Similarly,

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots \qquad \text{(ii)}$$

Adding eq. (i) and eq. (ii) by neglecting higher derivatives, we get,

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x)$$

$$\boxed{\therefore f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}}$$

Example

Q. Find approximation to second derivative of cos(x) at x=0.75 with h=0.01.

Sol^n:

We have,

$$f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

$$\therefore f''(0.75) = \frac{f(0.76)-2f(0.75)+f(0.74)}{0.0001} = -0.000305$$

Write down algorithm and C program for differentiating continuous function using three point formula [CSIT, 2077]

The three-point formula is a numerical method for approximating the derivative of a continuous function. It uses three points on the function to estimate the slope at a given point. Here's an algorithm for differentiating a continuous function using the three-point formula:

Algorithm:

1. Define the function to be differentiated.
2. Determine the step size (h) for selecting the three points.
3. Take an input for the point at which the derivative is to be calculated (x).
4. Calculate the derivative using the three-point formula: a. Let x1 = x - h b. Let x2 = x c. Let x3 = x + h d. Calculate f(x1), f(x2), and f(x3) using the given function. e. Calculate the derivative using the formula: derivative = (f(x3) - f(x1)) / (2 * h).
5. Print the calculated derivative.

```c
#include <stdio.h>
#include <math.h>

// Function to differentiate
double function(double x) {
    return sin(x);  // Example: Differentiating sin(x)
}

// Function to calculate derivative using three-point formula
double differentiate(double x, double h) {
    double x1 = x - h;
    double x2 = x;
    double x3 = x + h;
    double y1 = function(x1);
    double y2 = function(x2);
    double y3 = function(x3);

    double derivative = (y3 - y1) / (2 * h);
    return derivative;
}
```

```c
int main() {
    double x, h;
    printf("Enter the point at which the derivative is to be calculated: ");
    scanf("%lf", &x);
    printf("Enter the step size (h): ");
    scanf("%lf", &h);

    double derivative = differentiate(x, h);
    printf("The derivative at x = %lf is %lf\n", x, derivative);

    return 0;
}
```

Derivative Using Newton Forward Interpolation Formula

Newton forward interpolation formula is,

$$y(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots \dots\dots\dots$$

Where, $s = \frac{x-x_0}{h} \Rightarrow x = x_0 + sh$

$$y(x_0 + sh) = y_0 + s\Delta y_0 + \frac{s^2-s}{2!}\Delta^2 y_0 + \frac{s^3-3s^2+2s}{3!}\Delta^3 y_0 + \cdots \dots\dots\dots$$

Differentiating w.r.to. 's' we get,

$$y'(x) = \frac{dy}{dx}h = \Delta y_0 + \frac{(2s-1)}{2!}\Delta^2 y_0 + \frac{3s^2-6s+2}{3!}\Delta^3 y_0 + \cdots \dots\dots\dots$$

$$\therefore \frac{dy}{dx} = y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2s-1)}{2!}\Delta^2 y_0 + \frac{3s^2-6s+2}{3!}\Delta^3 y_0 + \cdots \dots\dots\right]$$

Differentiating Again, we get,

$$\frac{d^2y}{dx^2} = y''(x) = \frac{1}{h^2}[\Delta^2 y_0 + (s-1)\Delta^3 y_0 + \cdots \dots\dots]$$

Example

***Q.** The distance travelled by a car at various time intervals are given as follows:*

t(sec)	*1.5*	*2.0*	*2.5*	*3.0*	*3.5*	*4.0*
y(meter)	*3.375*	*7.0*	*13.625*	*24*	*38.875*	*59*

Evaluate the velocity and acceleration of the car at t=1.5 sec.

Sol^n:

The forward difference table for given data

t	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375					
		3.625				
2.0	7.0		3.0			
		6.625		0.75		
2.5	13.625		3.75		0	
		10.375		0.75		0
3.0	24		4.5		0	
		14.875		0.75		
3.5	38.875		5.25			
		20.125				
4.0	59					

Here,

$x_0 = 1.5$, x =1.5, h = 0.5, $s = \frac{x-x_0}{h} = \frac{1.5-1.5}{0.5} = 0$

We have,

$$\frac{dy}{dx} = y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2s-1)}{2!}\Delta^2 y_0 + \frac{3s^2-6s+2}{3!}\Delta^3 y_0 + \cdots \dots \dots\right]$$

$$\therefore \left(\frac{dy}{dx}\right)_{1.5} = y'(1.5) = \frac{1}{0.5}\left[3.625 + \left(-\frac{1}{2}\right) \times 3 + \frac{2}{6}(0.75)\right]$$

= 4.75 m/sec

Again,

$$\frac{d^2y}{dx^2} = y''(x) = \frac{1}{h^2}[\Delta^2 y_0 + (s-1)\Delta^3 y_0 + \cdots \dots \dots]$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{1.5} = y''(1.5) = \frac{1}{0.5^2}[3.0 + (-1) \times 0.75]$$

=9 m/sce^2

Numerical Integration

The process of evaluating a definite integral from a set of tabulated value of the integral $f(x)$ is called numerical integration.

$$I = \int_{x_0}^{x_n} f(x)dx$$

Newton's cotes formula

Let $y = f(x)$ be a function and $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots\ldots\ldots\ldots\ldots, y_n = f(x_n)$.
Where, $x_n = x_0 + nh$
By Newton forward interpolation formula,

$$f(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots \ldots\ldots\ldots\ldots$$

Where, $s = \frac{x-x_0}{h} \Rightarrow x = x_0 + sh$

Now,

$$\int_{x_0}^{x_n} ydx = \int_{x_0}^{x_0+nh} f(x)dx$$

Put $x = x_0 + sh \Rightarrow dx = h\,ds$ $\Rightarrow s = 0 \text{ to } s = n$

$$= \int_0^n f(x_0 + sh)h\,ds$$

$$\int_{x_0}^{x_n} y dx = h \int_0^n \left[y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots \ldots \right] ds$$

$$= h \int_0^n \left[y_0 + s\Delta y_0 + \left(\frac{s^2-s}{2}\right)\Delta^2 y_0 + \left(\frac{s^3-3s^2+2s}{6}\right)\Delta^3 y_0 + \cdots \ldots \right] ds$$

$$= h \left[sy_0 + \frac{s^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{s^3}{3} - \frac{s^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{s^4}{4} - \frac{3s^3}{3} + \frac{2s^2}{2}\right)\Delta^3 y_0 + \cdots \ldots \right]_0^n$$

$$= h \left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{12}(2n^3 - 3n^2)\Delta^2 y_0 + \frac{1}{24}(n^4 - 4n^3 + 4n^2)\Delta^3 y_0 + \cdots \ldots \right]$$

$$= nh \left[y_0 + \frac{n}{2}\Delta y_0 + \frac{1}{12}(2n^2 - 3n)\Delta^2 y_0 + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 y_0 + \cdots \ldots \right]$$

$$\therefore \int_{x_0}^{x_n} y dx = nh \left[y_0 + \frac{n}{2}\Delta y_0 + \frac{1}{12}(2n^2 - 3n)\Delta^2 y_0 + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 y_0 + \cdots \ldots \right]$$

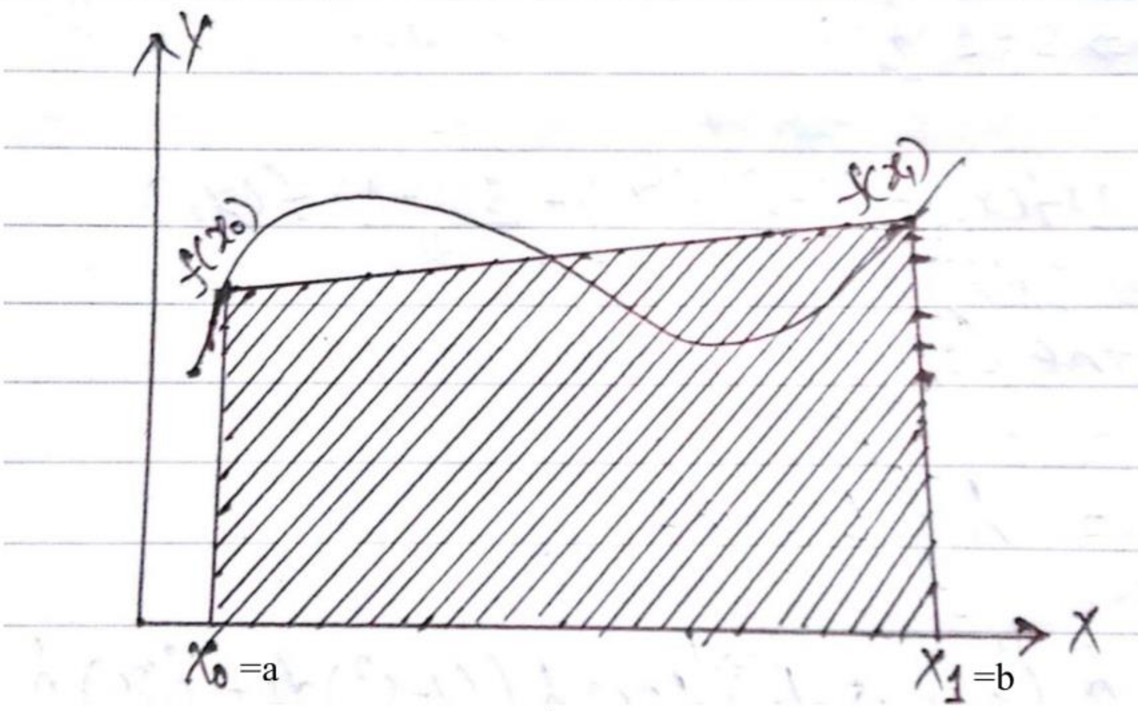
OR

$$\int_{x_0}^{x_n} f(x) dx = nh \left[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{1}{12}(2n^2 - 3n)\Delta^2 f(x_0) + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 f(x_0) + \cdots \ldots \right]$$

This is called **Newton's cotes formula**.

Trapezoidal Rule

Let $f(x)$ be a function and $x_0 = a$ and $x_1 = b$.



$h = x_1 - x_0$

As we know from Newton's cotes formula,

$$\int_{x_0}^{x_n} f(x)dx = nh\left[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{1}{12}(2n^2 - 3n)\Delta^2 f(x_0) + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 f(x_0) + \cdots \ldots \ldots\right] \quad \ldots\ldots\ldots\ldots\ldots (i)$$

By putting $n = 1$ in eq. (i) and neglecting higher term we get,

$$\int_{x_0}^{x_1} f(x)dx = h\left[f(x_0) + \frac{1}{2}\Delta f(x_0)\right]$$

$$= h\left[f(x_0) + \frac{1}{2}[f(x_1) - f(x_0)]\right]$$

$$= h\left[f(x_0) + \frac{1}{2}f(x_1) - \frac{1}{2}f(x_0)\right]$$

$$= \frac{h}{2}[f(x_0) + f(x_1)]$$

$$\therefore \int_{x_0}^{x_1} f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)]$$

This equation is called *trapezoidal rule.*

The Trapezoidal rule is a numerical integration method used to approximate the definite integral of a function. It approximates the area under a curve by dividing it into a series of trapezoids and summing up their areas.

The basic idea of the Trapezoidal rule is to approximate the curve of the function over a given interval by a series of straight line segments. Each segment is formed by connecting two adjacent points on the curve, and the area under each segment is calculated as the area of a trapezoid.

Example

Q. ***Evaluate the integral*** $\boldsymbol{I = \int_1^2 (x^3 + 1)dx}$***. Using trapezoidal rule.***

Soln:

Here,

$a = 1,\ b = 2,$

We have,

$$I = \frac{b-a}{2}[f(a) + f(b)]$$

$$= \frac{2-1}{2}[2 + 9]$$

$$= 5.5$$

Composite Trapezoidal Rule

If the range is to be integrated is large, the trapezoidal rule can be improved by dividing the interval (a, b) into the small intervals and applying the rule discussed above to each of these subintervals. The sum of the subintervals is the integral of the interval (a, b).

In the fig. below, there are $n + 1$ equally spaced sampling points that creates n segments of equal width h given by,

$$h = \frac{b-a}{n} \qquad x_i = a + ih, \qquad i = 0, 1, 2, \ldots \ldots .., n$$

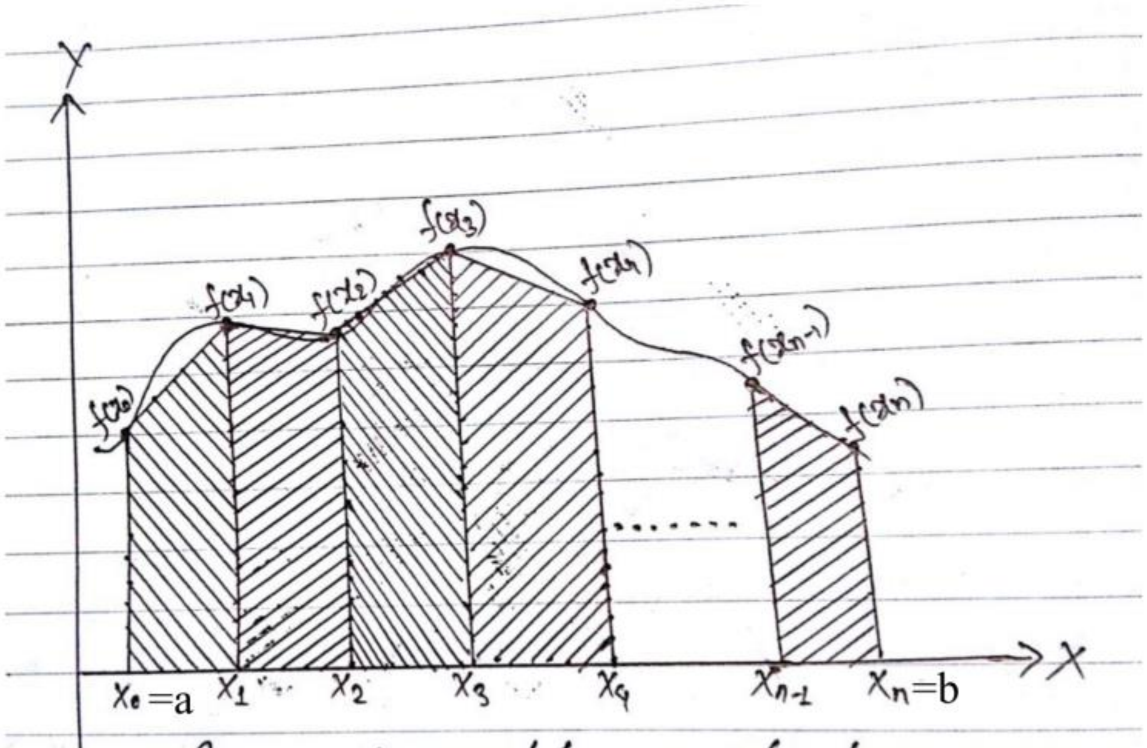


fig: Multisegment trapezoidal rule

From trapezoidal rule, area of the subinterval with the nodes x_{i-1} and x_i is given by

$$I = \frac{h}{2}[f(x_{i-1}) + f(x_i)]$$

The total area of all the n segments is,

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + \dots\dots\dots.. + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \frac{h}{2}[f(x_2) + f(x_3)] \dots\dots\dots.. + \frac{h}{2}[f(x_{n-1}) + f(x_n)]$$

$$= \frac{h}{2}[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \cdots \dots.. + f(x_{n-1}) + f(x_n)]$$

$$= \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots \dots.. + 2f(x_{n-1}) + f(x_n)]$$

$$= \frac{h}{2}[f(x_0) + 2\sum_i^{n-1} f(x_i) + f(x_n)]$$

$$\therefore \int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2\sum_i^{n-1} f(x_i) + f(x_n)]$$

Which is the *composite trapezoidal rule.*

Algorithm for trapezoidal rule

1. *Start*
2. *Read the value of lower limit of integration (a), upper limit of integration (b) and number of segments (n).*
3. *Compute* $h = (b - a)/n$

   $x=a$
4. *for i=0 to n*

   $y_i = f(x)$

   $x = x + h$

   Repeat i
5. *compute sum=* y_0+y_n
6. *for i=1 to n-1*

   *sum+=2** y_i

   Repeat i
7. *Compute sum=(h/2)* sum*
8. *Display sum as integral value.*
9. *END*

Examples

1. Compute the integral $\int_0^1 \frac{1}{1+x^2}\,dx$ using trapezoidal rule when n=5.

Here,

$a = 0$, $b = 1$, $n = 5$ so,

$h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$

$f(x) = \frac{1}{1+x^2}$

So we get the following table,

x_i	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$f(x_i)$	1	0.9615	0.8620	0.7352	0.6097	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

By trapezoidal rule; we have

$$I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5]$$

$$= \frac{1}{10}[1 + 2(0.9615 + 0.8620 + 0.7352 + 0.6097) + 0.5]$$

$$= 0.78368$$

$$\therefore \int_0^1 \frac{1}{1+x^2}\,dx = 0.78368$$

2. *Evaluate $\int_0^1 e^{-x^2}\,dx$ using trapezoidal rule with n=10.*

Here,

$a = 0$, $b = 1$, $n = 10$ so,

$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$

$f(x) = e^{-x^2}$

So we get the following table,

x_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x_i)$	1	0.99	0.96	0.91	0.85	0.77	0.69	0.61	0.52	0.44	0.36

By trapezoidal rule,

$$\int_0^1 e^{-x^2}\,dx = \frac{0.1}{2}[1 + 2(0.99 + 0.96 + 0.91 + 0.85 + 0.77 + 0.69 + 0.61 + 0.52 + 0.44) + 0.36]$$

$$\therefore \int_0^1 e^{-x^2}\,dx = 0.74$$

3. *Evaluate $\int_0^1 \sqrt{sinx + cosx}\ dx$ by trapezoidal rule with $h = 0.2$.*

Sol**^{n}**:

Here,

$a = 0$, $b = 1$, $h = 0.2$ so,

$n = \frac{b-a}{h} = \frac{1-0}{0.2} = 5$

$f(x) = \sqrt{sinx + cosx}$

So we get the following table,

x_i	0	0.2	0.4	0.6	0.8	1
$f(x_i)$	1	1.0857	1.448	1.1789	1.1891	1.1755

By trapezoidal rule,

$$\int_0^1 \sqrt{sinx + cosx}\ dx = \frac{0.2}{2}[1 + 2(1.0857 + 1.448 + 1.1789 + 1.1891) + 1.1755]$$

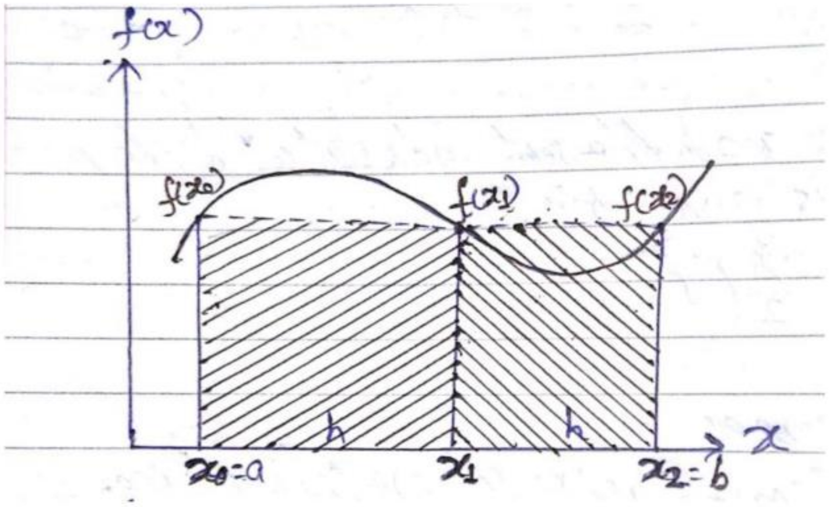
$$= 1.19789$$

$\therefore \int_0^1 \sqrt{sinx + cosx}\ dx = 1.19789$

➢ Simpson's 1/3 Rule

Here, the function $f(x)$ is approximated by a second-order polynomial $p_2(x)$ which passes through three sampling points as shown in fig. The three points include the end points $x_0(=a)$ and $x_2(=b)$ and a midpoint between them i.e. $x_0 = a$ and $x_2 = b$ and $x_1 = (a+b)/2$. The width of the segments h is given by

$$h = \frac{b-a}{2}$$



As we know from Newton's cotes formula,

$$\int_{x_0}^{x_n} f(x)dx = nh\left[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{1}{12}(2n^2 - 3n)\Delta^2 f(x_0) + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 f(x_0) + \cdots \ldots \ldots\right] \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

By putting $n = 2$ in eq. (i) and neglecting higher term we get,

$$\int_{x_0}^{x_2} f(x)dx = 2h\left[f(x_0) + \Delta f(x_0) + \frac{1}{6}\Delta^2 f(x_0)\right]$$

$$= 2h\left[f(x_0) + [f(x_1) - f(x_0)] + \frac{1}{6}[\Delta f(x_1) - \Delta f(x_0)]\right]$$

$$= 2h\left[f(x_1) + \frac{1}{6}[\{f(x_2) - f(x_1)\} - \{f(x_1) - f(x_0)\}]\right]$$

$$= 2h\left[f(x_1) + \frac{1}{6}[f(x_2) - 2f(x_1) + f(x_0)]\right]$$

$$= 2h\left[f(x_1) + \frac{1}{6}f(x_2) - \frac{1}{3}f(x_1) + \frac{1}{6}f(x_0)\right]$$

$$= 2h\left[\frac{1}{6}f(x_0) + \frac{2}{3}f(x_1) + \frac{1}{6}f(x_2)\right]$$

$$= \frac{2h}{6}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$\boxed{\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

Q. ***Evaluate the integral*** $\int_0^{\frac{\pi}{2}} \sqrt{sinx}\, dx$ ***using Simpson's 1/3 rule.***

Here,

a = 0, b = $\frac{\pi}{2}$ so,

$$h = \frac{b-a}{2} = \frac{\frac{\pi}{2}-0}{2} = \frac{\pi}{4}$$

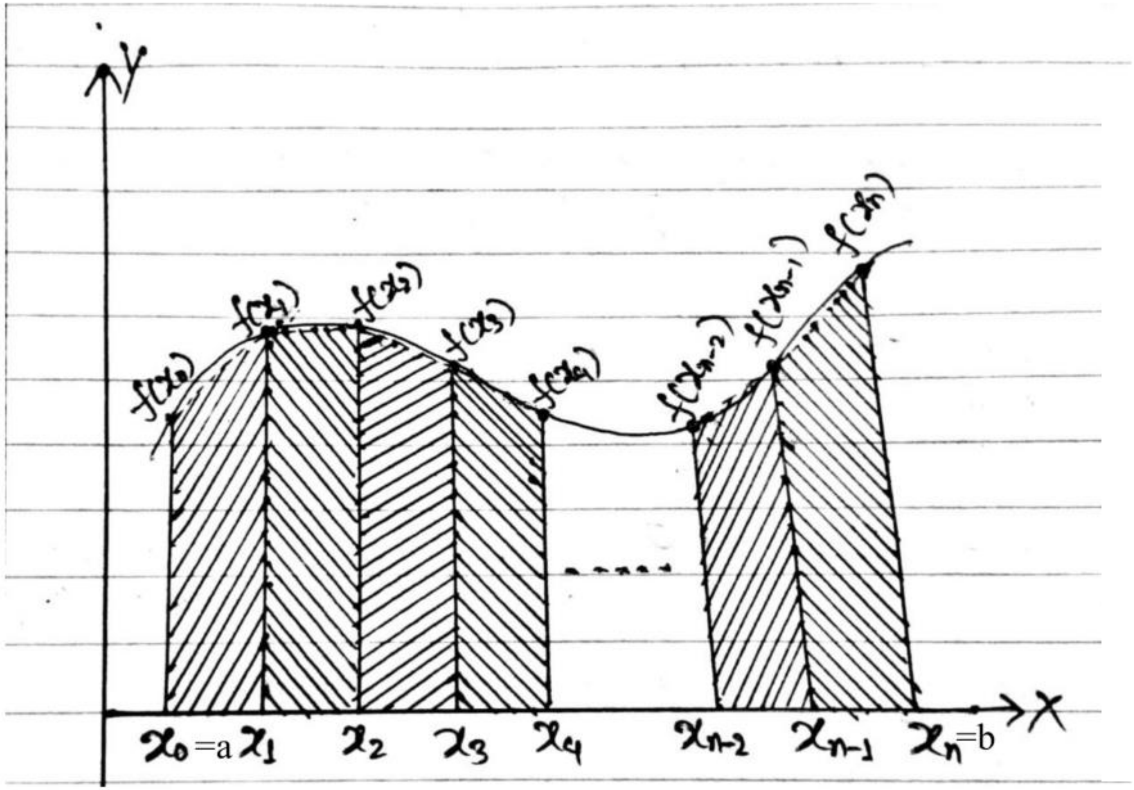
We have, $I = \frac{h}{3}[f(a) + 4f(x_1) + f(b)]$

$$x_1 = \frac{a+b}{2} = \frac{0+\frac{\pi}{2}}{2} = \frac{\pi}{4}$$

$$\therefore\ I = \frac{\pi}{12}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right]$$

$$= 0.26179(0 + 3.3637 + 1) \quad = \ 1.143$$

Composite Simpson's 1/3 Rule



$$\int_a^b f(x)dx = \int_a^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots \dots\dots\dots.. + \int_{x_{n-2}}^{b} f(x)dx$$

$$= \frac{h}{3}[f(a) + 4f(x_1) + f(x_2)] + \frac{h}{3}[f(x_2) + 4f(x_3) + f(x_4)] + \dots\dots\dots\dots\dots\dots\dots + \frac{h}{3}[f(x_{n-2}) + 4f(x_{n-1}) + f(b)]$$

$$= \frac{h}{3}[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \dots.. + 4f(x_{2i-1}) + 2f(x_{2i}) + f(b)]$$

$$= \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{\left(\frac{n}{2}\right)-1} f(x_{2i}) + f(b)\right]$$

$$\boxed{\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{\left(\frac{n}{2}\right)-1} f(x_{2i}) + f(b)\right]}$$

This equation is called *composite Simpson's 1/3 rule.*

Here, the integration interval is divided into *n* number of segments of equal width, where *n* is even number. Then the step size is

$$h = \frac{b-a}{n}$$

Examples

1. Compute the integral $\int_0^1 \frac{1}{1+x^2} dx$ ***applying Simpson's 1/3 rule using n=5.***

Sol:

Here, $h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$

$a = 0$, $b = 1$, $n = 5$ so, $f(x) = \frac{1}{1+x^2}$

So we get the following table,

x_i	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$f(x_i)$	1	0.9615	0.8620	0.7352	0.6097	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

By Simpson's 1/3 rule; we have,

$$I = \frac{h}{3}[y_0 + 2(y_2 + y_4) + 4(y_1 + y_3) + y_5]$$

$$= \frac{1}{15}[1 + 2(0.8620 + 0.6097) + 4(0.9615 + 0.7352) + 0.5]$$

$$= 0.74863$$

$$\therefore \int_0^1 \frac{1}{1+x^2} dx = 0.74863$$

2. Using Simpson's 1/3 rule evaluate $\int_0^2 (e^{x^2} - 1)\,dx$ with $n = 8$.

Sol**^{n}**:

Here,

$a = 0$, $b = 2$, $n = 8$ so,

$h = \frac{b-a}{n} = \frac{2-0}{8} = 0.25$

$f(x) = e^{x^2} - 1$

So we get the following table,

x_i	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0
$f(x_i)$	0	0.0645	0.2840	0.7550	1.7183	3.7707	8.4877	20.3809	53.5981
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's 1/3 rule; we have,

$I = \frac{h}{3}[y_0 + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7) + y_8]$

$= \frac{0.25}{3}[0 + 2(0.2840 + 1.7183 + 8.4877) + 4(0.0645 + 0.7550 + 3.7707 + 20.3809) + 53.5981]$

$= 14.5385$

$\therefore \int_0^2 (e^{x^2} - 1)\,dx = 14.5385$

3. *Using Simpson's 1/3 rule evaluate $\int_{0.2}^{1.2}(x^2 + lnx - sinx)\,dx$. (Take h=0.1)*

Sol^n:

Here,

$a = 0.2$, $b = 1.2$, $h = 0.1$

$f(x) = (x^2 + lnx - sinx)$

So we get the following table,

x_i	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
$f(x_i)$	-1.768	-1.409	-1.146	-0.922	-0.715	-0.511	-0.300	-0.079	0.158	0.414	0.690

By Simpson's 1/3 rule; we have,

$$\int_{0.2}^{1.2}(x^2 + lnx - sinx)\,dx = \frac{0.1}{3}[-1.768 + 2(-1.146 - 0.715 - 0.300 + 0.158) + 4(-1.409 - 0.922 - 0.511 - 0.079 + 0.414) + 0.690]$$

$= -0.5037$

$$\therefore \int_{0.2}^{1.2}(x^2 + lnx - sinx)\,dx = -0.5037$$

Simpson's 3/8 Rule

By putting $n = 3$ in newton's cotes formula and applying the same procedure followed in trapezoidal or Simpson's 1/3 rule, we can show that

$$\int_a^b f(x) = \frac{3h}{8}[f(a) + 3f(x_1) + 3f(x_2) + f(b)]$$

Where, $h = \frac{b-a}{3}$. This equation is called *Simpson's 3/8 rule*.

Example

Q. Use Simpson's 3/8 rule to evaluate $\int_1^2 (x^3 + 1)dx$.

Sol^n:

Here,

$a = 1$, $b = 2$

$h = \frac{b-a}{3} = \frac{2-1}{3} = \frac{1}{3}$

$f(x) = (x^3 + 1)$

We have,

$\int_a^b f(x) = \frac{3h}{8}[f(a) + 3f(x_1) + 3f(x_2) + f(b)]$

$x_1 = a + h = 1 + \frac{1}{3} = \frac{4}{3}$

$x_2 = a + 2h = 1 + \frac{2}{3} = \frac{5}{3}$

$\therefore I = \int_1^2 f(x) = \frac{1}{8}[f(1) + 3f(4/3) + 3f(5/3) + f(2)]$

$= 4.75$

Composite Simpson's 3/8 Rule

$$\int_a^b f(x) = \frac{3h}{8}[f(a) + 3[f(x_1) + f(x_2) + f(x_4) + f(x_5) + \cdots \ldots \ldots . + f(x_{n-1})] + 2[f(x_3) + f(x_6) + \cdots \ldots \ldots + f(x_{n-3})] + f(b)]$$

Examples

1. Integrate the function $\int_0^{\frac{\pi}{2}} \frac{sinx}{x} dx$ ***by Simpson's 3/8 rule. Take n=5.***

Here,

a = 0, b = $\frac{\pi}{2}$, n = 5 so,

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{5} = \frac{\pi}{10}$$

$$f(x) = \frac{sinx}{x}$$

So we get the following table,

x_i	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{\pi}{2}$
$f(x_i)$	0	0.017453	0.017452	0.017451	0.017451	0.017451
	y_0	y_1	y_2	y_3	y_4	y_5

By Simpson's 3/8 rule;

$$\int_0^{\frac{\pi}{2}} \frac{sinx}{x} dx = \frac{3h}{8}[y_0 + 2y_3 + 3(y_1 + y_2 + y_4) + y_5]$$

$$= \frac{3\pi}{80}[0 + 2 \times 0.017451 + 3(0.017453 + 0.017452 + 0.017451) + 0.017451]$$

= 0.02467

2. Compute the integral $\int_0^6 \frac{1}{1+x^2}\,dx$ applying Simpson's 3/8 rule using n=6.

Here,

a= 0, b=6 , n=6 so,

$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

$f(x) = \frac{1}{1+x^2}$

So we get the following table,

x_i	0	1	2	3	4	5	6
$f(x_i)$	1	0.5	0.2	0.1	0.059	0.038	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule;

$$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{8}[y_0 + 2y_3 + 3(y_1 + y_2 + y_4 + y_5) + y_6]$$

$$= \frac{3\text{x}1}{8}[1 + 2y_3 + 3(0.5 + 0.2 + 0.059 + 0.038) + 0.027]$$

$= 1.3571$

So, What is the difference between Trapezoidal Rule and Simpon’s 1/3 rule?

Simpson's 1/3 rule and the trapezoidal rule are both numerical methods used for approximating definite integrals. However, there are significant differences between them in terms of their approach and accuracy.

1.Approach:

1. Simpson's 1/3 Rule: This method approximates the function over each subinterval with a quadratic polynomial. It assumes that the function can be well-approximated by a smooth curve and uses a quadratic interpolation technique to estimate the integral.
2. Trapezoidal Rule: This method approximates the function over each subinterval with a straight line segment connecting the endpoints of the subinterval. It assumes that the function can be well-approximated by a straight line and uses linear interpolation to estimate the integral.

2.Accuracy:

1. Simpson's 1/3 Rule: This rule tends to provide a more accurate approximation compared to the trapezoidal rule for functions that are relatively smooth and can be well-approximated by quadratic polynomials. It can yield more accurate results since it takes into account the curvature of the function within each subinterval.
2. Trapezoidal Rule: This rule is less accurate than Simpson's 1/3 rule but still provides reasonable approximations. It works well for functions that are less smooth or have sharp discontinuities since it approximates the function by straight line segments.

1. Number of Function Evaluations:
   1. Simpson's 1/3 Rule: This rule requires function evaluations at the endpoints and the midpoint of each subinterval. Therefore, it needs an even number of intervals to work correctly.
   2. Trapezoidal Rule: This rule requires function evaluations only at the endpoints of each subinterval. It can work with any number of intervals, even or odd.

2. Error Convergence:
   1. Simpson's 1/3 Rule: The error of Simpson's 1/3 rule decreases as the number of intervals increases. The error convergence is generally faster compared to the trapezoidal rule.
   2. Trapezoidal Rule: The error of the trapezoidal rule also decreases as the number of intervals increases, but the convergence rate is slower compared to Simpson's 1/3 rule.

In summary, Simpson's 1/3 rule provides a more accurate approximation of definite integrals than the trapezoidal rule, particularly for smooth functions. However, the choice between the two methods depends on the characteristics of the function being integrated and the desired level of accuracy.

- **Romberg Integration**

The Romberg integration method uses the trapezoidal rule. It use trapezoidal rule in iterative way.

Formula to find Romberg integration

Use trapezoidal rule to find

I_1 = Divide the interval into two parts

I_2 = Divide the interval into four parts

I_3 = Divide the interval into eight parts

Then find,

$I_4 = \frac{4I_2 - I_1}{3}$

$I_5 = \frac{4I_3 - I_2}{3}$

Now, final result

$I = \frac{4I_5 - I_4}{3}$

Examples

1. Use Romberg integration method to evaluate integration $\int_4^{5.2} \log x\, dx$.

We have $h = \frac{b-a}{2} = \frac{5.2-4}{2} = 0.6$

Therefore, taking $h = 0.6$, $\frac{0.6}{2} = 0.3$ & $\frac{0.3}{2} = 0.15$

Let us calculate the given integral using trapezoidal rule.

$$f(x) = \log x$$

i) Taking $h = 0.6$

x_i	4	4.6	5.2
$f(x_i)$	0.60206	0.662758	0.716003

$I_1 = \frac{0.6}{2}[0.60206 + 2 \times 0.662758 + 0.716003] = 0.79$

ii) Taking $h = 0.3$

x_i	4	4.3	4.6	4.9	5.2
$f(x_i)$	0.60206	0.633468	0.662758	0.690196	0.716003

$I_2 = \frac{0.3}{2}[0.60206 + 2(0.633468 + 0.662758 + 0.690196) + 0.716003] = 0.794$

iii) Taking $h = 0.15$

x_i	4	4.15	4.3	4.45	4.6	4.75	4.9	5.04	5.2
$f(x_i)$	0.60206	0.618048	0.633468	0.64836	0.662758	0.67669	0.690196	0.70243	0.716003

$I_3 = \frac{0.15}{2}[0.60206 + 2(0.618048 + 0.633468 + 0.64836 + 0.662758 + 0.67669 +$
$0.70243 + 0.690196) + 0.716003] = 0.793$

Now,

$$I_4 = \frac{4I_2 - I_1}{3} = \frac{4\times0.794 - 0.793}{3} = 0.794$$

$$I_5 = \frac{4I_3 - I_2}{3} = \frac{4\times0.793 - 0.794}{3} = 0.792$$

$$I = \frac{4I_5 - I_4}{3} = \frac{4\times0.792 - 0.794}{3} = 0.791$$

$$\therefore \int_4^{5.2} \log x\, dx = \mathbf{0.791}$$

2. *Use Romberg integration method to evaluate integration* $\int_0^1 \frac{1}{1+x}\,dx$.

We have $h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$

Therefore, taking $h = 0.5,\ \frac{0.5}{2} = 0.25\ \&\ \frac{0.25}{2} = 0.125$

Let us calculate the given integral using trapezoidal rule.

$$f(x) = \frac{1}{1+x}$$

i) Taking $h = 0.5$

x_i	0	0.5	1
$f(x_i)$	1	0.667	0.5

$I_1 = \frac{0.5}{2}[1 + 2 \times 0.667 + 0.5] = 0.708$

ii) Taking $h = 0.25$

x_i	0	0.25	0.5	0.75	1
$f(x_i)$	1	0.8	0.667	0.5714	0.5

$I_2 = \frac{0.25}{2}[1 + 2(0.8 + 0.667 + 0.5714) + 0.5] = 0.696$

iii) Taking $h = 0.125$

x_i	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x_i)$	1	0.889	0.8	0.7273	0.667	0.615	0.5714	0.533	0.5

$I_3 = \frac{0.125}{2}[1 + 2(0.889 + 0.8 + 0.7273 + 0.667 + 0.615 + 0.5714 + 0.533) + 0.5] = 0.6941$

Now,

$I_4 = \frac{4I_2 - I_1}{3} = \frac{4\times0.696 - 0.708}{3} = 0.693$

$I_5 = \frac{4I_3 - I_2}{3} = \frac{4\times0.6941\ - 0.696}{3} = 0.693$

$I = \frac{4I_5 - I_4}{3} = \frac{4\times0.693\ - 0.693}{3} = 0.693$

$\therefore\ \int_0^1 \frac{1}{1+x}\,dx = \mathbf{0.693}$

Lab Questions

8. Write a C program to integrate a function using Trapazoidal Rule
9. Write a C program to integrate a function using Simpon’s 1/3 rule
10. Write a C program to integrate a function using Simpon’s 3/8 rule

3. (a) For the function $f(x) = x^2 e^{\sqrt{x}} \sin x$ estimate f'(2.1) and f''(2.7) [take h = 0.05] (3) asked in 2075(Old Course)

(b) Evaluate $\int_0^1 \left(e^{-x} + x \tan x\right) dx$ using trapezoidal rule taking h = 0.1 and h = 0.2. Also improve your result using Romberg integration.(4)

3. Derive Composite Simpson's 1/3 Rule for numerical integration. How does it improve the accuracy of integration? show solution asked in Model Question

8. Estimate the first derivative of *f(x)=ln x* at *x=1* using the second order central difference formula. show solution asked in Model Question

9. Evaluate the following integration using Romberg integration. asked in 2078

$$\int_0^1 \frac{\sin^2 x}{x} dx$$

8. Calculate the integral value of the function given below from x = 1.8 to x = 3.4 using Simpson's 1/3 rule. asked in 2075(New Course)

X	1.8	2.0	2.2	2.4	2.6	2.8	3.0	3.4
F(x)	0.003	0.778	1.632	2.566	3.579	4.672	7.097	8.429

8. Write down algorithm and program for the differentiating continuous function using three point formula. asked in 2077

8. Write down algorithm and program for the differentiating continuous function using three point formula. asked in 2077

9. Evaluate the following integration using Romberg integration. asked in 2075(New Course)

$$\int_0^1 \frac{\sin x}{x} dx .$$

9. How Simpson's 1/3 rule differs from trapezoidal rule? Derive the formula for Simpson's 1/3 rule. asked in 2077

3. Derive the composite formula for the trapezoidal rule with its geometrical figure. Evaluate $I = \int_0^1 e^{-x^2}$ dx using this rule with n=5, upto 6 decimal places.(4 + 4) asked in 2066

3. Write Newton-cotes integration formulas in basic form for x = 1, 2, 3 and give their composite rules. Evaluate $I = \int_{.2}^{1.5} e^{-x^2}$ using the Gaussian integration three point formula. (4+4) asked in 2067

3. Evaluate $I = \int_0^1 e^{-x^2}$ using trapezoidal rule with n=10. Also evaluate the same integral using Grossion 3 point formula and compare the result. (4+4) asked in 2068

3. a) For the function $f(x) = e^x \sqrt{sinx + lnx}$ estimate f'(6.3) and f"(6.3) [take h = 0.01] (4) asked in 2069

b) Evaluate $\int_1^2 (lnx + x^2 \ sinx)dx$ using Gaussian integration 3 point formula. (4)

3. Derive Simpson's 1/3 rule to evaluate numerical integration. Using this formula evaluate (4 + 4) asked in 2070

$\int_{0.2}^{1.2} (x^2 + lnx - sinx)dx$. [Take h = 0.1]

5. Write a program to calculate the integral using Trapezoidal Rules?

10 (a) Fit a straight line to the following set of data points.

X	1	3	4	6	8	9
y	1	3	4	4	5	7