Solution of Ordinary Differential Equations

UNIT 5 | [8 hrs]

Unit 5 Solution of Ordinary Differential Equations **7 Hrs.**

Introduction to Differential Equations, Initial Value Problem, Taylor Series Method, Picard's Method, Euler's Method and Its Accuracy, Heun's method,

Runge-Kutta Methods, Solution of Higher Order Equations, Boundary Value Problems, Shooting Method and Its Algorithm.

Solution of Ordinary Differential Equations

Let x be an independent variable and y be a dependent variable. An equation with x, y and its derivatives is called a differential equation.

Suppose the first order differential equation;

$$\frac{dy}{dx} = f(x, y) \quad \dots\dots\dots\dots \quad (i)$$

A solution to the differential equation is the value of y which satisfies the differential equation.

Initial Value Problem

Consider the differential equation

$y' = f(x, y)$ with an initial condition $y(x_0) = y_0$.

This is the first order differential equation. Here the y value at x_0 is given to be y_0. The solution y at x_0 is given.

We must assume a small increment h.

$$x_1 = x_0 + h$$

$$x_2 = x_1 + h$$

$$\dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots$$

$$x_{i+1} = x_i + h$$

Let us denote the y values at $x_1, x_2, \dots\dots$ as $y_1, y_2, \dots\dots$ respectively.

y_0 is given and we must find out $y_1, y_2, \dots\dots$

The initial value y_0 is given. So, this differential equation is called an ***initial value problem.***

Boundary Value Problem

Consider the following linear second order differential equation,

$$y'' + f(x)y' + g(x)y = F(x)$$

Suppose we are interested in solving this differential equation between the values $x = a$ & $z = b$. Hence a & b are two values such that $a < b$. Let us divide the interval $[a, b]$ into n equal subintervals of length h each.

Let $x_0, x_1, \dots \dots \dots, x_n$ be the pivotal points and b are called the boundary points. Solving the differential equation means finding the values of $y_0, y_1, \dots \dots \dots, y_n$.

Suppose y_0 & y_n are given. That is the solution values at the boundary points are given. Then the differential equation is called a **Boundary Value Problem**. So, the following is the general form of a boundary value problem.

$$y'' + f(x)y' + g(x)y = F(x)$$

$$y(a) = y_0\ , \quad y(b) = y_n$$

Boundary Value Problem

In numerical methods, a boundary value problem (BVP) refers to a type of differential equation problem that involves finding the solution to a differential equation subject to specified conditions at the boundaries of the domain. Unlike initial value problems (IVPs) that require initial conditions at a single point, boundary value problems require conditions at multiple points.

Let's consider the following second-order ordinary differential equation with boundary conditions:

$$y''(x) - 4y'(x) + 4y(x) = 0$$

with boundary conditions:

$$y(0) = 1$$
$$y(2) = 4$$

To solve this boundary value problem, we need to find the function $y(x)$ that satisfies the differential equation and the boundary conditions.

One approach to solving BVPs numerically is the shooting method. In the shooting method, we transform the BVP into an IVP by guessing an initial value for the derivative $y'(a)$ at the left boundary $x = a$ (in this case, $x = 0$), and then integrating the differential equation from $x = a$ to $x = b$ using a numerical integration method like the Runge-Kutta method. We then adjust the initial guess for $y'(a)$ iteratively until the value of $y(b)$ matches the right boundary condition $y(b) = 4$.

Aspect	Initial Value Problem (IVP)	Boundary Value Problem (BVP)
Definition	A differential equation with an initial condition.	A differential equation with boundary conditions.
Conditions	One condition at a specific point (initial point).	Multiple conditions at different boundary points.
Solution Type	Typically, unique solutions (may be generalizable).	Generally, not unique solutions (can have none, one, or multiple).
Nature of Solution	Usually requires only first-order derivatives.	Can require higher-order derivatives.
Existence of Solution	Often guaranteed under certain conditions (e.g., existence and uniqueness theorem).	Not always guaranteed; existence depends on the problem's nature.
Method of Solution	Often solved using numerical methods (e.g., Euler's method, Runge-Kutta).	May involve analytical methods or numerical techniques (e.g., Finite Element Method).
Example	Newton's second law of motion (acceleration as a function of time).	Heat conduction equation (temperature distribution with boundary temperatures).

➢ Taylor's Series Method

y is a function of x. It is written as $y(x)$. By Taylor's series about the point x_0;

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots \dots \dots$$

x_0 & y_0 denote the initial value of x & y.

Examples

1. ***Find by Taylor's series method, the values of*** $\boldsymbol{y}$ ***at*** $\boldsymbol{x = 0.1}$ & $\boldsymbol{x = 0.2}$ ***to fine places of decimal form.***

$$\boldsymbol{\frac{dy}{dx} = x^2 y - 1; \quad y(0) = 1}$$

Solution:

Given,

$$\frac{dy}{dx} = x^2y - 1$$

$$y(0) = 1$$

i.e. $x_0 = 0$ & $y_0 = 1$

Here,

$$y' = x^2y - 1$$

$$y'' = x^2y' + 2xy$$

$$y''' = x^2y'' + 2xy' + 2(xy' + y) = x^2y'' + 4xy' + 2y$$

$$y^{iv} = x^2y''' + 6xy'' + 6y'$$

Now at $x_0 = 0$ & $y_0 = 1$;

$$y_0' = x_0^2y_0 - 1 = 0 - 1 = -1$$

$$y_0'' = x_0^2y_0' + 2x_0y_0 = 0 + 0 = 0$$

$$y_0''' = x_0^2y_0'' + 4x_0y_0' + 2y_0 = 0 + 0 + 2 \times 1 = 2$$

$$y_0^{iv} = x_0^2y_0''' + 6x_0y_0'' + 6y_0' = 0 + 0 + 6 \times (-1) = -6$$

Now, the Taylor's series is;

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y_0'''' \text{ [Neglecting higher term]}$$

$$= 1 + \frac{x}{1!} \times (-1) + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 2 + \frac{x^4}{4!} \times (-6)$$

$$= 1 - \frac{x}{1!} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$$

$$\therefore y(0.1) = 1 - \frac{0.1}{1!} + \frac{2\times 0.1^3}{3!} - \frac{6\times 0.1^4}{4!} = 0.900308$$

$$\therefore y(0.2) = 1 - \frac{0.2}{1!} + \frac{2\times 0.2^3}{3!} - \frac{6\times 0.2^4}{4!} = 0.802267$$

2. ***Find the solution of following differential equation using Taylor's series method.***

$$\boldsymbol{y' = (x^3 + xy^2)e^{(-x)},\ y(0) = 1 \text{ to find } y \text{ at } x = 0.1, 0.2, 0.3}$$

Sol^n:

Given,

$$y' = (x^3 + xy^2)e^{(-x)}$$

$$y(0) = 1 \text{ i.e. } x_0 = 0 \ \&\ y_0 = 1$$

Here

$$y' = (x^3 + xy^2)e^{(-x)}$$

$$\begin{aligned} y'' &= (x^3 + xy^2)(-e^{-x}) + e^{-x}(3x^2 + x2yy' + y^2) \\ &= e^{-x}(3x^2 + x2yy' + y^2 - x^3 - xy^2) \\ &= e^{-x}(3x^2 - x^3 + y^2 + 2xyy' - xy^2) \end{aligned}$$

$$\begin{aligned} y'' &= (x^3 + xy^2)(-e^{-x}) + e^{-x}(3x^2 + x2yy' + y^2) \\ &= e^{-x}(3x^2 + x2yy' + y^2 - x^3 - xy^2) \\ &= e^{-x}(3x^2 - x^3 + y^2 + 2xyy' - xy^2) \end{aligned}$$

$$\begin{aligned} y''' &= e^{-x}[6x - 3x^2 + 2yy' + 2\{x(yy'' + (y')^2) + yy'\} - (x2yy' + y^2)] - e^{-x}(3x^2 - x^3 + y^2 + 2xyy' - xy^2) \\ &= e^{-x}[6x - 3x^2 + 2yy' + 2x(yy'' + (y')^2) + 2yy' - 2xyy' - y^2 - 3x^2 + x^3 - y^2 - 2xyy' + xy^2] \\ &= e^{-x}[x^3 - 6x^2 + 6x + 4yy' + 2x(yy'' + (y')^2) - 4xyy' - 2y^2 + xy^2] \end{aligned}$$

Now at $x_0 = 0 \ \&\ y_0 = 1$;

$$y_0' = 0$$

$$y_0'' = 1(0 - 0 + 1 + 0 - 0) = 1$$

$$y_0''' = 1(0 - 0 + 0 + 0 + 0 - 0 - 2 + 0) = -2$$

Now, the Taylor's series is;

$$y(x) = y_0 + \frac{(x-x_0)}{1!}\, y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' \qquad \text{[Neglecting higher term]}$$

$$= 1 + \frac{(x-0)}{1!}\,(0) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(-2)$$

$$= 1 + \frac{x^2}{2} - \frac{x^3}{3}$$

$$\therefore\ y(0.1) = 1 + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{3} = 1.0047$$

$$\therefore\ y(0.2) = 1 + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{3} = 1.0173$$

$$\therefore\ y(0.3) = 1 + \frac{(0.3)^2}{2} - \frac{(0.3)^3}{3} = 1.036$$

3. *Use the Taylor method to solve the equation*

$$\boldsymbol{y' = x^2 + y^2}$$

for $\boldsymbol{x = 0.25}$ ***and*** $\boldsymbol{x = 0.5}$ ***given*** $\boldsymbol{y(0) = 1}$**.**

Sol

Given,

$$y' = x^2 + y^2$$

$$y(0) = 1 \text{ i.e. } x_0 = 0 \;\&\; y_0 = 1$$

Here,

$$y' = x^2 + y^2$$

$$y'' = 2x + 2yy'$$

$$y''' = 2 + 2yy'' + 2(y')^2$$

Now at $x_0 = 0 \;\&\; y_0 = 1$;

$$y_0' = 1$$

$$y_0'' = 0 + 2 = 2$$

$$y_0''' = 2 + 4 + 2 = 8$$

[Neglecting higher term]

Now, the Taylor's series is;

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0'''$$

$$= 1 + \frac{(x-0)}{1!}(1) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{3!}(8)$$

$$= 1 + x + x^2 + \frac{8x^3}{3!}$$

$$\therefore y(0.25) = 1 + 0.25 + (0.25)^2 + \frac{8(0.25)^3}{3!} = 1.3333$$

$$\therefore y(0.25) = 1 + 0.5 + (0.5)^2 + \frac{8(0.5)^3}{3!} = 1.81667$$

- **Picard's Method**

Consider the differential equation;

$$\frac{dy}{dx} = f(x, y) \ \ldots\ldots\ldots\ (i)$$

with given initial condition $y(x_0) = y_0$

Eq.(i) can be written as

$$dy = f(x, y)dx \ \ldots\ldots\ldots\ \text{(ii)}$$

Integrating eq.(ii) from x_0 to x w.r.to x.

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx$$

$$[y]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)dx$$

$$y - y_0 = \int_{x_0}^{x} f(x, y)dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx$$

For 1st approximation we replace y by y_0 we get,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

For 2nd approximation we replace y by y_1 we get,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

Similarly, for other approximation we make a general form;

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})dx$$

We continue this process until we get two successive approximation value equal.

1. Obtain a solution up to the fifth approximation of the equation $\frac{dy}{dx} = y + x$ ***such that*** $y(0) = 1$ ***using Picard's process of successive approximation.***

Solution:

Here,

$$\frac{dy}{dx} = y + x$$

$$y(0) = 1$$

i.e. $x_0 = 0$ & $y_0 = 1$

Using Picard's formula, we have;

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})dx$$

1st approximation;

$$y_1 = y_0 + \int_0^x f(x, y_0)dx$$
$$y_1 = 1 + \int_0^x (y_0 + x)dx$$
$$y_1 = 1 + \int_0^x (y_0 + x)dx$$
$$y_1 = 1 + \int_0^x (1 + x)dx$$
$$y_1 = 1 + x + \frac{x^2}{2}$$

2nd approximation;

$$y_2 = y_0 + \int_0^x f(x, y_1)dx$$
$$y_2 = 1 + \int_0^x (y_1 + x)dx$$
$$y_2 = 1 + \int_0^x (1 + 2x + \frac{x^2}{2})dx$$
$$y_2 = 1 + x + x^2 + \frac{x^3}{6}$$

3rd approximation;

$$y_3 = y_0 + \int_0^x f(x, y_2)dx$$
$$y_3 = 1 + \int_0^x (y_2 + x)dx$$
$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{6})dx$$
$$y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

4th approximation;

$$y_4 = y_0 + \int_0^x f(x, y_3)dx$$
$$y_4 = 1 + \int_0^x (y_3 + x)dx$$
$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24})dx$$
$$y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

5th approximation;

$$y_5 = y_0 + \int_0^x f(x, y_4)dx$$
$$y_4 = 1 + \int_0^x (y_4 + x)dx$$
$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120})dx$$
$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120})dx$$
$$y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

2. ***Use Picard's method, estimate y(0.1) of the following equation;***

$$\boldsymbol{y'(x) = x^2 + y^2, \quad y(0) = 0}$$

Solution:

Here,

$y'(x) = x^2 + y^2$

$y(0) = 0$

i.e. $x_0 = 0$ & $y_0 = 0$

Using Picard's formula, we have;

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})dx$$

First approximation;

$y_1 = y_0 + \int_0^x f(x, y_0)dx$

$y_1 = 0 + \int_0^x x^2 dx = \frac{x^3}{3}$

$At\ x = 0.1,$

$y_1 = 0.00033$

Second approximation,

$y_2 = y_0 + \int_0^x f(x, y_1)dx$

$y_2 = 0 + \int_0^x f(x, y_1)dx = \frac{x^3}{3} + \frac{x^7}{63}$

$At\ x = 0.1,$

$y_2 = 0.00033$

Here, $y_1 = y_2$ up to 5 decimal places.

$\therefore y(0.1) = 0.00033$

➢ Euler's Method

In Euler's method, the slope at (x_i, y_i) is used to estimate the value of $y(x_{i+1})$ as below;

$$y(x_{i+1}) = y(x_i) + m_1 h \; ; \; m_1 = f(x_i, y_i)$$

Choosing smaller values of h leads to more accurate results and more computation time.

Algorithm:

1. *Define* $f(x, y)$.
2. *Read* x_0, y_0, h *and* xn *where* x_0& y_0 *are initial conditions,* h *is the interval and* xp *is the required value.*
3. $n = \frac{xp - x_0}{h}$
4. *Start loop from* $i = 1$ *to* n
5. $y = y_0 + h * f(x_0, y_0)$
   $x = x + h$
6. *Print values of* y_0 & x_0.
7. *Check if* $x < xp$
   assign $x_0 = x$ *and* $y_0 = y$
   else
   goto 8.
8. *End loop* i
9. *Stop*

Examples

1. Given $y' = xy$, $y(1) = 1$. Find $y(2)$ with $h = 0.25$.

Solution:

Here,
$y' = f(x, y) = xy$

$y(1) = 1$ **i.e.** $x_0 = 1$ & $y_0 = 1$

Then,
$y(1) = y_0 = 1$

$y(1.25) = y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0 * y_0) = 1 + 0.25 * (1 * 1) = 1.25$

$y(1.5) = y_2 = y_1 + hf(x_1, y_1) = 1.25 + 0.25 * (1.25 * 1.25) = 1.64$

$y(1.75) = y_3 = y_2 + hf(x_2, y_2) = 1.64 + 0.25 * (1.5 * 1.64) = 2.26$

$y(2) = y_4 = y_3 + hf(x_3, y_3) = 2.26 + 0.25 * (1.75 * 2.26) = 3.25$

Hence,
$y(2) = 3.25$

2. *Given the equation* $\boldsymbol{y' = 2x^3 - 3xy,\ y(1) = 2}$***. Find*** $\boldsymbol{y(2.5)}$ ***with*** $\boldsymbol{h = 0.5}$**.**

<u>Solution:</u>

Here,

$$y' = f(x, y) = 2x^3 - 3xy$$

$y(1) = 2$ **i.e.** $x_0 = 1$ & $y_0 = 2$

Then,

$$y(1) = y_0 = 2$$

$$y(1.5) = y_1 = y_0 + hf(x_0, y_0) = 2 + 0.5[2 - 3(1)(2)] = 0$$

$$y(2) = y_2 = y_1 + hf(x_1, y_1) = 0 + 0.5[2 * 1.5^3 - 3 * 1.5 * 0] = 3.375$$

$$y(2.5) = y_3 = y_2 + hf(x_2, y_2) = 3.375 + 0.5[2 * 2^3 - 3 * 2 * 3.375] = 1.25$$

Hence,

$$y(2.5) = 1.25$$

➢ **Heun's Method**

- This method is also called ***second order Runge-Kutta method*** or ***Modified Euler's method.***

In Heun's method, we use the average of the slopes computed at the beginning and at the end of the interval.

Using Heun's method, we can estimate the value of $y(x_{i+1})$ as below;

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}(m_1 + m_2) \qquad // \ y(x_{i+1}) = \ y(x_i + h)$$

Where, $m_1 = f(x_i, y_i)$

$$m_2 = f(x_i + h, y_i + m_1 \times h)$$

Algorithm:

1. *Define* $f(x, y)$.
2. *Read* x_0, y_0, h *and* n
3. *For i=0 to n-1 do*
4. $x_{i+1} = x_i + h$
5. $m_1 = f(x_i, y_i)$
6. $m_2 = f(x_i + h, y_i + m_1 \times h)$
7. $y_{i+1} = y_i + \frac{h}{2}(m_1 + m_2)$
8. *Print* x_{i+1} , y_{i+1}
9. *Next i*
10. *End*

1. ***Use the Heun's method to estimate*** $\boldsymbol{y(0.4)}$ ***when*** $\boldsymbol{y'(x) = x^2 + y^2}$ ***with*** $\boldsymbol{y(0) = 0}$***.***
Assume $\boldsymbol{h = 0.2}$***.***

Here,
$y'(x) = f(x, y) = x^2 + y^2$
$y(0) = 0$ **i.e.** $x_0 = 0$ & $y_0 = 0$
$h = 0.2$
From Heun's method, we have;

<u>1st iteration:</u>

$$m_1 = f(x_0, y_0) = x_0^2 + y_0^2 = 0 + 0 = 0$$
$$m_2 = f(x_0 + h, y_0 + m_1 * h) = f(0 + 0.2, 0 + 0 * 0.2) = f(0.2, 0) = 0.2^2 + 0^2 = 0.04$$
$$\therefore y(x_0 + h) = y(x_0) + \frac{h}{2}(m_1 + m_2)$$
$$y(0 + 0.2) = y(0.2) = y(0) + \frac{0.2}{2}(m_1 + m_2) = 0 + \frac{0.2}{2}(0 + 0.04) = 0.004$$
$$\therefore y(0.2) = 0.004$$

<u>2nd iteration:</u>

Here,
$x_1 = 0.2$ & $y_1 = 0.004$

$$m_1 = f(x_1, y_1) = x_1^2 + y_1^2 = 0.2^2 + 0.004^2 = 0.040016$$
$$m_2 = f(x_1 + h, y_1 + m_1 * h) = f(0.4, 0.012) = 0.4^2 + 0.012^2 = 0.160144$$
$$\therefore y(x_1 + h) = y(x_1) + \frac{h}{2}(m_1 + m_2)$$
$$y(0.2 + 0.2) = y(0.4) = y(0.2) + \frac{0.2}{2}(0.040016 + 0.160144) = 0.004 + 0.02 = 0.024$$

$$\boldsymbol{\therefore y(0.4) = 0.024}$$

2. *Apply Runge Kutta method of 2nd order to find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y(0) = 1$.*

Here,

$\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{x} + \mathrm{y}$

$\mathrm{y}(0) = 1$ i.e. $x_0 = 0$ & $y_0 = 1$

let us assume $h = 0.2$

$$m_1 = f(x_0, y_0) = f(0, 1) = 0 + 1 = 1$$

$$m_2 = f(x_0 + h, y_0 + m_1 \times h) = f(0 + 0.2, 1 + 1 \times 0.2) = f(0.2, 1.2) = 1.4$$

Then,

$$y(0.2) = y(0) + \frac{0.2}{2}(m_1 + m_2) = 1 + \frac{0.2}{2}(1 + 1.4) = 1.24$$

$$\therefore y(0.2) = 1.24$$

Fourth Order Runge-Kutta (R-K) Method

$$y_{i+1} = y_i + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Where,

$$m_1 = f(x_i, y_i)$$
$$m_2 = f(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2})$$
$$m_3 = f(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2})$$
$$m_4 = f(x_i + h, y_i + m_3 h)$$

$$y_1 = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Where,

$$m_1 = f(x_0, y_0)$$
$$m_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2})$$
$$m_3 = f(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2})$$
$$m_4 = f(x_0 + h, y_0 + m_3 h)$$

Similarly, for second interval

$$y_2 = y_1 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Where,

$$m_1 = f(x_1, y_1)$$
$$m_2 = f(x_1 + \frac{h}{2}, y_1 + \frac{m_1 h}{2})$$
$$m_3 = f(x_1 + \frac{h}{2}, y_1 + \frac{m_2 h}{2})$$
$$m_4 = f(x_1 + h, y_1 + m_3 h)$$

Algorithm:

1. *Define* $f(x, y)$.
2. *Read* x_0, y_0, h *and* n
3. *For i=0 to n-1 do*
4. $x_{i+1} = x_i + h$
5. $m_1 = f(x_i, y_i)$
6. $m_2 = f(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2})$
7. $m_3 = f(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2})$
8. $m_4 = f(x_i + h, y_i + m_3 h)$
9. $y_{i+1} = y_i + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$
10. *Print* x_{i+1} , y_{i+1}
11. *Next i*
12. *End*

***1.** Apply Runge Kutta method of 4^{th} order to find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y(0) = 1$.*

Here,

$$\frac{dy}{dx} = f(x, y) = x + y$$

$y(0) = 1$ *i.e.* $x_0 = 0$ & $y_0 = 1$

let us assume $h = 0.2$

Now, from Runge-Kutta method, we have,

$$m_1 = f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1$$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right) = f\left(0 + \frac{0.2}{2}, 1 + \frac{1 \times 0.2}{2}\right) = f(0.1, 1.1) = 1.2$$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right) = f\left(0 + \frac{0.2}{2}, 1 + \frac{1.2 \times 0.2}{2}\right) = f(0.1, 1.12) = 1.22$$

$$m_4 = f(x_0 + h, y_0 + m_3 h) = f(0 + 0.2, 1 + 1.22 \times 0.2) = f(0.2, 1.244) = 1.444$$

Hence,

$$y_1 = y(x_0 + h) = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$\therefore y(0.2) = 1 + \frac{0.2}{6}(1 + 2 \times 1.2 + 2 \times 1.22 + 1.444)$$

$$= 1.2428$$

2. *Obtain y(1.5) from given differential equation using Runge-Kutta 4th order method.*

$$\frac{dy}{dx} + 2x^2y = 1 \text{ and } y(1) = 0. \qquad [\text{take } h=0.25]$$

Here,

$f(x,y) = \frac{dy}{dx} = 1 - 2x^2y$

$y(1) = 0$ i.e. $x_0 = 1$ & $y_0 = 0$

$h=0.25$

Hence,

$$y_1 = y(x_0 + h) = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$\therefore y(1.25) = 0 + \frac{0.25}{6}(1 + 2 \times 0.684 + 2 \times 0.784 + 0.388)$$
$$= 0.18$$

Now, from Runge-Kutta method, we have,

1st iteration

$$m_1 = f(x_0, y_0) = 1 - 2x_0^2 y_0 = 1 - 2 \times 1^2 \times 0 = 1 - 0 = 1$$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right) = f\left(1 + \frac{0.25}{2}, 0 + \frac{1 \times 0.25}{2}\right) = f(1.125, 0.125) = 0.684$$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right) = f\left(1 + \frac{0.25}{2}, 0 + \frac{0.684 \times 0.25}{2}\right) = f(1.125, 0.0855) = 0.784$$

$$m_4 = f(x_0 + h, y_0 + m_3 h) = f(1 + 0.25, 0 + 0.784 \times 0.25) = f(1.25, 0.196) = 0.388$$

2nd iteration

Now,

$x_1 = 1.25$ & $y_1 = 0.18$

$m_1 = f(x_1, y_1) = 1 - 2x_1^2 y_1 = 1 - 2 \times 1.25^2 \times 0.18 = 1 - 0.5625 = 0.437$

$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_1 h}{2}\right) = f\left(1.25 + \frac{0.25}{2}, 0.18 + \frac{0.437 \times 0.25}{2}\right) = f(1.375, 0.235) = 0.111$

$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_2 h}{2}\right) = f\left(1.25 + \frac{0.25}{2}, 0.18 + \frac{0.111 \times 0.25}{2}\right) = f(1.375, 0.194) = 0.266$

$m_4 = f(x_1 + h, y_1 + m_3 h) = f(1.25 + 0.25, 0.18 + 0.266 \times 0.25) = f(1.5, 0.246) =$ -0.107

Hence,

$y_2 = y(x_1 + h) = y_1 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$\therefore y(1.5) = 0.18 + \frac{0.25}{6}(0.437 + 2 \times 0.111 + 2 \times 0.266 + (-0.107))$

$= 0.2251$

$\therefore \boldsymbol{y(1.5) = 0.2251}$

Solving Higher Order Differential Equation

A high order differential equation is in the form

$$\frac{d^m y}{dx^m} = f\left(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \dots\dots\dots, \frac{d^{m-1} y}{dx^{m-1}}\right)$$

with m initial condition given as;

$y(x_0) = a_1$

$y'(x_0) = a_2$

. .

. .

. .

$y^{m-1}(x_0) = a_m$

Let us denote,

$y = y_1$

$\frac{dy}{dx} = y_2$

$\frac{d^2 y}{dx^2} = y_3$

. .

. .

. .

$\frac{d^{m-1} y}{dx^{m-1}} = y_m$

Then we can write,

$\frac{dy_1}{dx} = y_2$ with $y_1(x_0) = y_{10} = a_1$

$\frac{dy_2}{dx} = y_3$ with $y_2(x_0) = y_{20} = a_2$

.

.

.

$\frac{dy^{m-1}}{dx} = y_m$ with $y_{m-1}(x_0) = y_{(m-1)0} = a_{m-1}$

$\frac{dy^m}{dx} = \frac{d^m y}{dx^m} = F(x, y_1, y_2, \dots\dots\dots, y_m)$ with $y_m(x_0) = y_{m0} = a_m$

This system is similar to the system of first order equation.

Hence, we can solve this by any procedure applied for first order equation.

Representation of Higher Order Equation into Simultaneous Equation

Consider the seconder order differential equation

$$y'' = f(x, y, y')$$
$$y(x_0) = y_0, y'(x_0) = y_0'$$

This can be converted into a system of Simultaneous equations.

Put $z = y'$

Therefore the equation becomes $z' = f(x, y, z)$

$$y(x_0) = y_0$$

That is we have,

$$y' = z$$
$$z' = f(x, y, z)$$
$$y(x_0) = y_0, z(x_0) = y_0$$

This is a set of Simultaneous equations and hence can be solved. Any higher order equation can thus be transformed into simultaneous equation.

Q. ***Solve the following differential equation for y(0.5) using Heun's method.***

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2xy = 1 \text{ with } y(0) = 1 \text{ and } y'(0) = 1 \ .$$

Here,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2xy = 1$$

$$x_0 = 0 \ \& \ y_0 = 1, y'(0) = 1 = z_0$$

Put $\frac{dy}{dx} = z$ & differentiating w.r.to x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Equation assumes the form:

$$\frac{dz}{dx} + 3z + 2xy = 1$$

We have system of equations,

$$y' = \frac{dy}{dx} = z = f(x, y, z) \qquad \text{[let slope} = m_i\text{]}$$

$$\frac{dz}{dx} = 1 - 2xy - 3z = g(x, y, z) \qquad \text{[let slope} = l_i\text{]}$$

$let\ h = 0.5$

Now,

$$m_1 = f(x_0, y_0, z_0) = f(0,1,1) = 1$$

$$l_1 = g(x_0, y_0, z_0) = g(0,1,1) = 1 - 2 \times 0 \times 1 - 3 \times 1 = -2$$

Similarly,

$$m_2 = f(x_0 + h, y_0 + hm_1, z_0 + hl_1) = f(0 + 0.5, 1 + 0.5 \times 1, 1 + 0.5 \times (-2))$$

$$= f(0.5, 1.5, 0) = 0$$

$$l_2 = g(0.5, 1.5, 0) = 1 - 2 \times 0.5 \times 1.5 - 3 \times 0 = -0.5$$

$$\therefore y(0.5) = y_0 + \frac{h}{2}(m_1 + m_2)$$

$$= 1 + \frac{0.5}{2}(1 + 0)$$

$$= 1.25$$

$$\therefore y'(0.5) = y'(0) + \frac{h}{2}(l_1 + l_2)$$

$$= 1 + \frac{0.5}{2}(-2 - 0.5)$$

$$= 0.375$$

Q. ***Solve the following differential equation to find $y(0.1)$ using 4th order Runge-Kutta method.***

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1 \text{ with } y(0) = 1 \text{ and } y'(0) = 0$$

Here,

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$$

$$x_0 = 0 \;\&\; y_0 = 1, y'(0) = 0 = z_0$$

Put $\frac{dy}{dx} = z$ & differentiating w.r.to x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Equation assumes the form:

$$\frac{dz}{dx} - x^2z - 2xy = 1$$

We have system of equations,

$$y' = \frac{dy}{dx} = z = f(x, y, z) \qquad \text{[let slope} = m_i\text{]}$$

$$\frac{dz}{dx} = 1 + 2xy + x^2z = g(x, y, z) \qquad \text{[let slope} = l_i\text{]}$$

We have,

$y(x_0 + h) = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$ ……….. (i)

$x_0 + h = 0.1 \Longrightarrow h = 0.1$

$m_1 = f(x_0, y_0, z_0) = f(0, 1, 0) = 0$

$l_1 = g(x_0, y_0, z_0) = g(0, 1, 0) = 1 + 2 \times 0 \times 1 + 0^2 \times 0 = 1$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}, z_0 + \frac{l_1 h}{2}\right) = f(0.05,\ 1,\ 0.05) = 0.05$

$l_2 = g\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}, z_0 + \frac{l_1 h}{2}\right) = g(0.05,\ 1,\ 0.05) = 1.10$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}, z_0 + \frac{l_2 h}{2}\right) = f(0.05,\ 1.0025,\ 0.055) = 0.055$

$l_3 = g\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}, z_0 + \frac{l_2 h}{2}\right) = g(0.05,\ 1.0025,\ 0.055) = 1.1$

$m_4 = f(x_0 + h, y_0 + m_3 h, z_0 + l_3 h) = f(0.1, 1.005, 0.11) = 0.11$

$l_4 = g(x_0 + h, y_0 + m_3 h, z_0 + l_3 h) = g(0.1, 1.005, 0.11) = 1.202$

m_1, m_2, m_3 & m_4 values substituted in eq. (i)

$$y(0.1) = 1 + \frac{0.1}{6}(0 + 2 \times 0.05 + 2 \times 0.055 + 0.11)$$
$$= 1.0053$$

Boundary Value Problem

Consider the following linear second order differential equation,

$$y'' + f(x)y' + g(x)y = F(x)$$

Suppose we are interested in solving this differential equation between the values $x = a$ & $z = b$. Hence a & b are two values such that $a < b$. Let us divide the interval $[a, b]$ into n equal subintervals of length h each.

Let $x_0, x_1, \dots \dots \dots, x_n$ be the pivotal points and b are called the boundary points. Solving the differential equation means finding the values of $y_0, y_1, \dots \dots \dots, y_n$.

Suppose y_0 & y_n are given. That is the solution values at the boundary points are given. Then the differential equation is called a **Boundary Value Problem**. So, the following is the general form of a boundary value problem.

$$y'' + f(x)y' + g(x)y = F(x)$$

$$y(a) = y_0 \,, \quad y(b) = y_n$$

Shooting Method

... obtained. We hav...
... is achieved.

Algorithm

Start

Read Boundary conditions, say xa, xb, ya & yb

Read the point at which solution is needed, say xp

Read accuracy limit, say E

Convert higher order differential equation to system of d...

Read value of h

Approximate first approximation as below:

Set x=xa y=ya

g_1=(yb-ya)/(xb-xa)

Calculate y(xb) by using Euler's method

Set v_1=y

If(y<yb)

$g_2=2g_1$

else

$g_2=g_1/2$

Calculate y(xb) by using Euler's method

Set v_2=y

Compute new values of y(xb) as below

Compute $g_3 = g_2 - \dfrac{v_2 - y_b}{v_2 - v_1}(g_2 - g_1)$

Find $y(x_b)$ by using Euler's method

Compute error

if(error<E)

Display solution

Go to step 9

Else

Set $v_1=v_2$ v_2=y(xb)

Set $g_1=g_2$ $g_2=g_3$

Go to step 8

Terminate

Example

Solve the ordinary differential equation given below by using shooting method with Euler's method. And calculate the value of y(1.5) by using h=0.5.

$$\frac{d^2y}{dx^2} = 6x \qquad y(1) = 2 \qquad y(2) = 9$$

Let $\frac{dy}{dx} = z$

Then

$$\frac{dz}{dx} = 6x$$

This gives us two first order differential equations

$$\frac{dy}{dx} = z \qquad y(1) = 2$$

$$\frac{dz}{dx} = 6x \qquad z(1) = unknown$$

Let us assume

$$z(1) = \frac{y(2) - y(1)}{2-1} = 7$$

Now, set up the initial value problem as

$$\frac{dy}{dx} = z \qquad y(1) = 2$$

$$\frac{dz}{dx} = 6x \qquad z(1) = 7$$

Where,

$$f_1(x, y, z) = z$$

$$f_2(x, y, z) = 6x$$

From Euler's method, we know that

$$y_{i+1} = y_i + f_1(x_i, y_i, z_i)h$$

$$z_{i+1} = z_i + f_2(x_i, y_i, z_i)h$$

Calculate First Approximation

Iteration 1

$$x_0 = 1 \quad y_0 = 2 \quad z_0 = 7$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$

$$= 2 + f_1(1,2,7)h = 5.5$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$

$$= 7 + f_2(1,2,7)h = 10$$

Iteration 2

$$x_1 = 1.5 \quad y_1 = 5.5 \quad z_1 = 10$$

$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$

$$= 5.5 + f_1(1.5,5.5,10)h = 10.5$$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$

$$= 10 + f_2(1.5,5.5,10)h = 14.5$$

Thus, $y(2)=10.5$

The given value of this boundary condition is: $y(2)=9$.

Since, predicted value of $y\,(2)$ is higher than actual value

Let us assume

$$z(1) = \frac{1}{2} \times \frac{y(2) - y(1)}{2-1} = 3.5$$

Calculation of Second Approximation

Iteration 1

$$x_0 = 1 \quad y_0 = 2 \quad z_0 = 3.5$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$

$$= 2 + f_1(1,2,3.5)h = 3.75$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$

$$= 3.5 + f_2(1,2,3.5)h = 6.5$$

Iteration 2

$$x_1 = 1.5 \quad y_1 = 3.75 \quad z_1 = 6.5$$

$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$

$$= 3.75 + f_1(1.5,3.75,6.5)h = 7$$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$

$$= 6.5 + f_2(1.5,3.75,6.5)h = 11$$

Thus, $y(2)=7$

Since, Predicted values $y(1.5)$ is lower than actual value

Use linear interpolation on the previous guesses to obtain new guess as below:

$$g_3 = g_2 - \frac{v_2 - v}{v_2 - v_1}(g_2 - g_1)$$

$$= 3.5 - \frac{7-9}{7-10.5}(3.5-7) = 3.5 + 0.57 \times 3.5 \approx 5.5$$

Thus, new guess (g_3)=5.5.

Calculation of Third Approximation

Iteration 1

$$x_0 = 1 \quad y_0 = 2 \quad z_0 = 5.5$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$

$$= 2 + f_1(1,2,5.5)h = 4.75$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$

$$= 5.5 + f_2(1,2,5.5)h = 8.5$$

Iteration 2

$$x_1 = 1.5 \quad y_1 = 4.75 \quad z_1 = 8.5$$

$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$

≃ 4.75 ~~= 0~~ $+ f_1(1.5,4.75,8.5)h = 9$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$

$$= 8.5 + f_2(1.5,4.75,8.5)h = 13$$

Thus, $y(2)=9$

And the given value of this boundary condition is: $y(2)=9$.

Thus, we can use third approximation to obtain value $y(1.5)$

⇒ $y(1.5)=4.75$

Solve the ordinary differential equation given below by using shooting method with Euler's method. And calculate the value of y(3) and y(6) by using h=3.

$$\frac{d^2y}{dx^2} - 2y = 72x - 8x^2 \qquad y(0) = 0 \qquad y(9) = 0$$

Solution:

Let $\dfrac{dy}{dx} = z$

Then

$$\frac{dz}{dx} - 2y = 72x - 8x^2$$

This gives us two first order differential equations

$$\frac{dy}{dx} = z \qquad y(0) = 0$$

$$\frac{dz}{dx} = 2y + 72x - 8x^2 \qquad z(0) = unknown$$

Let us assume

$$z(0) = \frac{y(9) - y(0)}{9 - 0} = 0$$

Now, set up the initial value problem as

$$\frac{dy}{dx} = z \qquad y(0) = 0$$

$$\frac{dz}{dx} = 2y + 72x - 8x^2 \qquad z(0) = 0$$

Where,

$$f_1(x, y, z) = z$$

$$f_2(x, y, z) = 2y + 72x - 8x^2$$

From Euler's method, we know that

$$y_{i+1} = y_i + f_1(x_i, y_i, z_i)h$$

$$z_{i+1} = z_i + f_2(x_i, y_i, z_i)h$$

Calculate First Approximation

Iteration 1

$$x_0 = 0 \quad y_0 = 0 \quad z_0 = 0$$
$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$
$$= 0 + f_1(0,0,0)h = 0$$
$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$
$$= 0 + f_2(0,0,0)h = 0$$

Iteration 2

$$x_1 = 3 \quad y_1 = 0 \quad z_1 = 0$$
$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$
$$= 0 + f_1(3,0,0)h = 0$$
$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$
$$= 0 + f_2(3,0,0)h = 432$$

Iteration 3

$$x_2 = 6 \quad y_2 = 0 \quad z_2 = 432$$
$$y_3 = y_2 + f_1(x_2, y_2, z_2)h$$
$$= 0 + f_1(6,0,432)h = 1296$$
$$z_3 = z_2 + f_2(x_2, y_2, z_2)h$$
$$= 0 + f_2(6,0,432)h = 432$$

Thus, $y(9)=1296$

The given value of this boundary condition is: $y(9)=0$.

Since, predicted value of $y(9)$ is much higher than actual value

Assume that $z(0) = -10$

Calculation of Second Approximation

Iteration 1

$$x_0 = 0 \quad y_0 = 0 \quad z_0 = -10$$
$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$
$$= 0 + f_1(0,0,-10)h = -30$$
$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$
$$= 0 + f_2(0,0,-10)h = 0$$

Iteration 2

$$x_1 = 3 \quad y_1 = -30 \quad z_1 = 0$$
$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$
$$= 0 + f_1(3,-30,0)h = 0$$
$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$
$$= 0 + f_2(3,-30,0)h = 252$$

Iteration 3

$$x_2 = 6 \quad y_2 = 0 \quad z_2 = 252$$
$$y_3 = y_2 + f_1(x_2, y_2, z_2)h$$
$$= 0 + f_1(6,0,252)h = 756$$
$$z_3 = z_2 + f_2(x_2, y_2, z_2)h$$
$$= 0 + f_2(6,0,252)h = 432$$

Thus, $y(9)=756$

Since, Predicted values y(9) is much higher than actual value

Use linear interpolation on the previous guesses to obtain new guess as below:

$$g_3 = g_2 - \frac{v_2 - v}{v_2 - v_1}(g_2 - g_1)$$

$$= -10 - \frac{756 - 0}{756 - 1296}(-10 - 0) = -10 - 1.4 \times 10 = -24$$

Thus, new guess (g_3) = - 24

Calculation of Third Approximation

Iteration 1

$$x_0 = 0 \quad y_0 = 0 \quad z_0 = -24$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$

$$= 0 + f_1(0,0,-24)h = -72$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$

$$= 0 + f_2(0,0,-24)h = 0$$

Iteration 2

$$x_1 = 3 \quad y_1 = -72 \quad z_1 = 0$$

$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$

$$= 0 + f_1(3,-72,0)h = 0$$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$

$$= 0 + f_2(3,-72,0)h = 0$$

Iteration 3

$$x_2 = 6 \quad y_2 = 0 \quad z_2 = 0$$

$$y_3 = y_2 + f_1(x_2, y_2, z_2)h$$

$$= 0 + f_1(6,0,0)h = 0$$

$$z_3 = z_2 + f_2(x_2, y_2, z_2)h$$

$$= 0 + f_2(6,0,0)h = 432$$

Thus, *y(9)=0*

Thus, *y(9)=0*

And the given value of this boundary condition is: *y(9)=0*.

Thus, we can use third approximation to obtain value *y(3)* and *y(6)*

=> *y(3)=-72* *y(6)=0*

C program for boundary value problem using shooting method

```
#include <stdio.h>
#include <math.h>

#define MAX_ITERATIONS 1000
#define EPSILON 1e-6

// Define the function f(x, y, y')
double f(double x, double y, double yp) {
    // Replace this with your ODE function, for example:
    // return x * yp - y;
    // or any other ODE you want to solve.
}

// Shooting method to solve the boundary value problem
double shooting_method(double a, double b, double A, double
B) {
    double ya, yb, ypa, ypb, ym, yp, y;

    // Initial guesses for the derivatives at the boundaries
    double yp_low = 0.0;
    double yp_high = 1.0;
```

```
// Bisection method to find the correct initial condition
  for (int i = 0; i < MAX_ITERATIONS; i++) {
    y = ya = A;
    yp = ypa = yp_low;
    double mid = (yp_low + yp_high) / 2.0;

    // Numerical integration using Euler's method
    double h = (b - a) / 1000;
    for (double x = a; x < b; x += h) {
      ypb = yp;
      yp = yp + h * f(x, y, yp);
      y = y + h * ypb;
    }
    yb = y;
```

```
 // Check if the solution is close enough to the boundary condition B
        if (fabs(yb - B) < EPSILON) {
            return yb;
        }

        // Adjust the interval for bisection
        if (yb > B) {
            yp_high = mid;
        } else {
            yp_low = mid;
        }
    }
```

```
    // If the maximum number of iterations is reached, return an error value
    return NAN;
}

int main() {
    // Define the boundary conditions and the interval [a, b]
    double a = 0.0;
    double b = 1.0;
    double A = 0.0;
    double B = 1.0;

    // Solve the boundary value problem using the shooting method
    double yb = shooting_method(a, b, A, B);

    if (!isnan(yb)) {
        printf("The value of y(%lf) is approximately %lf\n", b, yb);
    } else {
        printf("Failed to converge to a solution.\n");
    }

    return 0;
}
```

3. What is higher order differential equation? How can you solve the higher order differential equation? Explain. Solve the following differential equation for $1 \leq x \leq 2$, taking *h=0.25*. asked in 2078

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = 0$$ with *y(1) = 1* and *y'(1) = 2*

3. What is initial value problem and boundary value problem? Write an algorithm and program to solve the boundary value problem using shooting method. asked in 2075(New Course)

5. (a) How can you solve higher order differential equation? Explain. (3) asked in 2075(Old Course)

(b) Solve the following differential equation within $1 \leq x \leq 2$ using Runge-Kutta 4^{th} order method. (5)

$$\frac{dy}{dx} + 3x - 4y = 2$$, with $y(1) = 1$. (Take $h = 0.25$).

5. How can you solve higher order differential equation? Explain. Solve the following differential within $0 \leq x \leq 1$ using Heun's method.(3 + 5) asked in 2069

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2xy = 1$$ with y(0)=1 and y'(0) = 1 (take h = 0.5)

5. Compare Euler's method with Heun's method for solving differential equation. Obtain y(1.5) from given differential equation using Runge-Kutta 4th order method.(4 + 4) asked in 2070

$$\frac{dy}{dx} + 2x^2y = 1$$ with y(1) = 0 (take h = 0.25)

Solve the following boundary value problem using shooting method.(8)

$\frac{d^2y}{dx^2} - 2x^2y = 1$, with y(0) = 1 and y(1) = 1 [Take h = 0.5].

5. Solve the following boundary value problem using shooting method.(8) asked in 2071

$\frac{d^2y}{dx^2} - 2x^2y = 1$, with y(0) = 1 and y(1) = 1 [Take h = 0.5].

5. Obtain y(1.5) to the following differential equation using Runge-Kutta 4th order method. (8) asked in 2073

$\frac{dy}{dx} + 2x^2y = 1$, with $y(1) = 0$ taking h = 0.25

5. How can you use Taylor series method to find the solution of ordinary differential equation? Use the Taylor method to solve the equation (5 + 3) $y' = x^2 + y^2$ for x = 0.25 and x = 0.5 given y(0) = 1 asked in 2074

OR

Define boundary value problem. Use shooting method to solve the equation $\frac{d^2y}{dx^2} = 6x$, $y(1) = 2$, $y(2) = 9$ in the interval(1, 2).

6. Explain the Picard's proves of successive approximation. Obtain a solution upto the fifth approximation of the equation $\frac{dy}{dx} = y + x$ such that y = 1 when x = 0 using Picard's process of successive approximations . (2+6) asked in 2067

6. Apply Runge-Kutta method of second order and fourth order to find an approximate value of y when x = 0.2 given that (8) asked in 2068

$\frac{\partial y}{\partial x} = x + y \text{ and } y(0) = 1.$

7. Write an algorithm and program for computer to obtain the solution of differential equation using Runge-Kutta Method. {5+7} asked in 2072

7. Define ordinary differential equation of the first order. What do you mean by initial value problem? Find by Taylor's series method, the values of y at x = 0.1 and x = 0.2 to find places of decimal form asked in 2066

$\frac{dy}{dx} = x^2y - 1$, when y(0) = 1 (2 + 6)

10. Explain about boundary value problem with example? Differentiate it with initial value problem

show solution
asked in Model Question

11. Use the Heun's method to estimate y(0.4) when

$y'(x)=x^2+y^2$ with y(0)=0. Assume h=0.2

show solution
asked in Model Question

11. Solve the following differential equation for $1 \le x \le 2$, taking $h = 0.25$ using Heun's method. asked in 2078

$y'(x) + x^2y = 3x$, with $y(1) = 1$

10. Appropriate the solution of y' = 2x + y , y(0) = 1 using Eulers method with step size 0.1. Approximate the value of y(0.4). asked in 2077

11. From the following differential equation estimate y(1) using RK 4^{th} order method. asked in 2075(New Course)

$\frac{dy}{dx} + 2x^2y = 4$ with y(0) = 1, [Take h = 0.5].

12. How boundary value problems differs from initial value problems? Discuss shooting method for solving boundary value problem. asked in 2077