Solutions of Partial Differential Equations

UNIT 6 [5 hrs]

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Solution of Partial Differential Equation

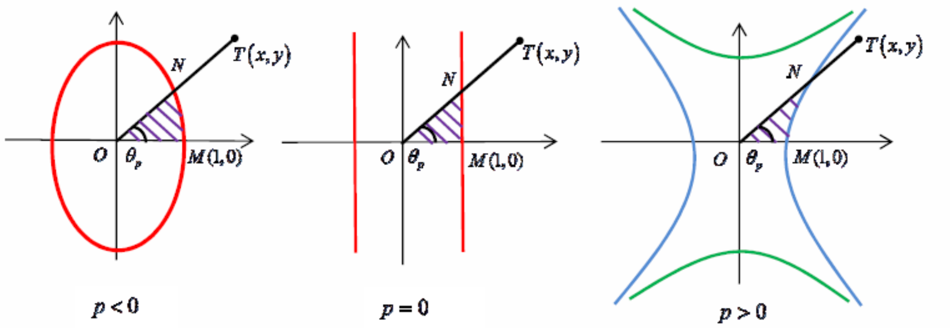
A partial differential equation is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variable), and a partial derivatives of the unknown function with respect to the independent variables.

If we represent the dependent variable as f and the two independent variables as x & y then second-order equation is given as

$$a\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial x\partial y}+c\frac{\partial^2 f}{\partial y^2}=F(x,y,f,\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})$$

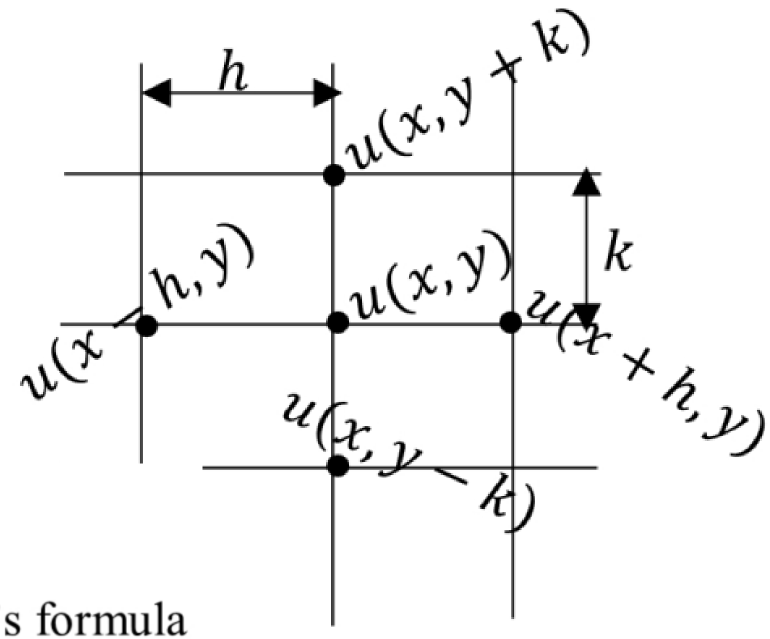
Where the coefficients a, b & c may be constants or functions of x & y. Depending upon the value of b^2-4ac , a 2nd order linear partial differential equation can be classified into three categories:

- Elliptical at point (x, y) if $b^2-4ac<0$
- Parabolic if $b^2-4ac=0$
- Hyperbolic if $b^2-4ac>0$



Deriving Difference Equations

Consider a two dimensional solution domain as shown in figure below;



The domain is split into regular rectangle grids of width h and height k.

Let $u(x,y)$ be the function of two independent variables x & y. Then by Taylor's formula

$$u(x+h,y) = u(x,y) + hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + \frac{h^3}{3!}u_{xxx}(x,y) + \cdots \text{.. .... (i)}$$

$$u(x-h,y) = u(x,y) - hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) - \frac{h^3}{3!}u_{xxx}(x,y) + \cdots \text{.. .... (ii)}$$

$$u(x,y+k) = u(x,y) + ku_y(x,y) + \frac{k^2}{2!}u_{yy}(x,y) + \frac{k^3}{3!}u_{yyy}(x,y) + \cdots \text{.. .... (iii)}$$

$$u(x,y-k) = u(x,y) - ku_y(x,y) + \frac{k^2}{2!}u_{yy}(x,y) - \frac{k^3}{3!}u_{yyy}(x,y) + \cdots \text{.. .... (iv)}$$

Subtracting eq.(ii) by (i) & neglecting higher power of h we get;

$$u(x+h,y) - u(x-h,y) = 2hu_x(x,y)$$

$$\boxed{\therefore u_x(x,y) = \frac{u(x+h,y) - u(x-h,y)}{2h}}$$

Subtracting eq.(iv) by (iii) & neglecting higher power of k we get;

$$u(x,y+k) - u(x,y-k) = 2ku_y(x,y)$$

$$\boxed{\therefore u_y(x,y) = \frac{u(x,y+k) - u(x,y-k)}{2k}}$$

Adding eq.(i) and (ii) and neglecting higher of power of h we get;

$$u(x+h,y)+\ u(x-h,y)=2u(x,y)+h^2u_{xx}(x,y)$$

$$\boxed{\therefore u_{xx}(x,y)=\frac{1}{h^2}[u(x+h,y)-2u(x,y)+\ u(x-h,y)]} \quad \text{(a)}$$

Adding eq.(iii) and (iv) and neglecting higher of power of k we get;

$$u(x,y+k)+\ u(x,y-k)=2u(x,y)+k^2u_{yy}(x,y)$$

$$\boxed{\therefore u_{yy}(x,y)=\frac{1}{k^2}[u(x,y+k)-2u(x,y)+\ u(x,y-k)]} \quad \text{(b)}$$

Also,

$$\boxed{u_{xy}(x,y)=\frac{u(x+h,y+k)-u(x+h,y-k)-u(x-h,y+k)+u(x-h,y-k)}{4hk}}$$

Laplace's Equation

The equation $\boldsymbol{u_{xx} + u_{yy} = 0}$ is the Laplace equation, then from above eq.(a) & (b) we have,

$$\frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)] + \frac{1}{k^2}[u(x,y+k) - 2u(x,y) + u(x,y-k)] = 0$$

If $h = k$ we get,

$$u(x+h,y) + u(x,y+k) + u(x-h,y) + u(x,y-k) - 4u(x,y) = 0$$

$$\boxed{\therefore\ u(x,y) = \frac{1}{4}[u(x+h,y) + u(x,y+k) + u(x-h,y) + u(x,y-k)]}$$

This is the difference equation for Laplace's equation.

Poisson's Equation

The equation $\boldsymbol{u_{xx} + u_{yy} = g(x,y)}$ is the given Poisson's equation, then from above eq.(a) & (b) we have,

$$\frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)] + \frac{1}{k^2}[u(x,y+k) - 2u(x,y) + u(x,y-k)] = g(x,y)$$

If $h = k$ we get,

$$\boxed{u(x+h,y) + u(x,y+k) + u(x-h,y) + u(x,y-k) - 4u(x,y) = h^2 g(x,y)}$$

This is the difference equation for Poisson's equation.

Examples

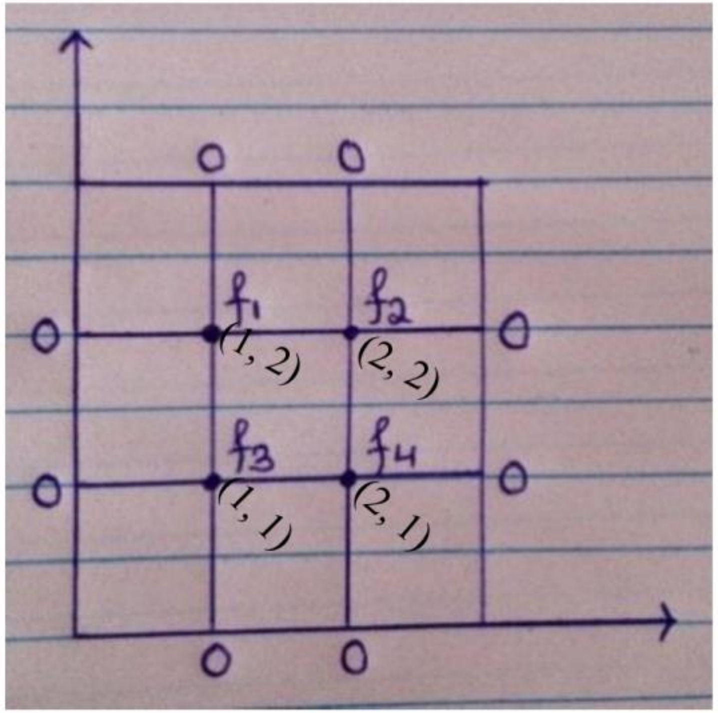
1. ***Solve the Poisson's equation $\nabla^2 f = 2x^2y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary and h=1.***

Solution:

Given Poisson's eqn. is $\nabla^2 f = 2x^2y^2$; $0 \le x \le 3, 0 \le y \le 3$

$h = 1$

Let's divide the domain into grids of 3×3 with $f = 0$ at the boundary as below



Now, from the difference equation for the Poisson's equation,

At f_1

$0 + 0 + f_2 + f_3 - 4f_1 = 1^2 \times 2 \times 1^2 \times 2^2$

Or, $f_2 + f_3 - 4f_1 = 8$ .... (i)

At f_2

$0 + 0 + f_1 + f_4 - 4f_2 = 1^2 \times 2 \times 2^2 \times 2^2$

Or, $f_1 + f_4 - 4f_2 = 32$ .... (ii)

At f_3

$0 + 0 + f_1 + f_4 - 4f_3 = 1^2 \times 2 \times 2^2 \times 1^2$

Or, $f_1 + f_4 - 4f_3 = 2$ .... (iii)

At f_4

$0 + 0 + f_2 + f_3 - 4f_4 = 1^2 \times 2 \times 1^2 \times 1^2$

Or, $f_2 + f_3 - 4f_4 = 8$ .... (iv)

Solving these equations,

Using eq.(ii) in (iii)

$32 - f_4 + 4f_2 + f_4 - 4f_3 = 2$

$4f_2 - 4f_3 = -30$ .... (a)

Using eq.(ii) in (iv)

$f_2 + f_3 - 4(32 - f_1 + 4f_2) = 8$

$f_2 + f_3 - 128 + 4f_1 - 16f_2 = 8$

$-15f_2 + f_3 + 4f_1 = 136$ .... (b)

& we have eq. (i)

$f_2 + f_3 - 4f_1 = 8$ …….. (i)

Solving equation (a), (b) & (i) we get,

$f_2 = -\frac{43}{4}$

$f_3 = -\frac{13}{4}$

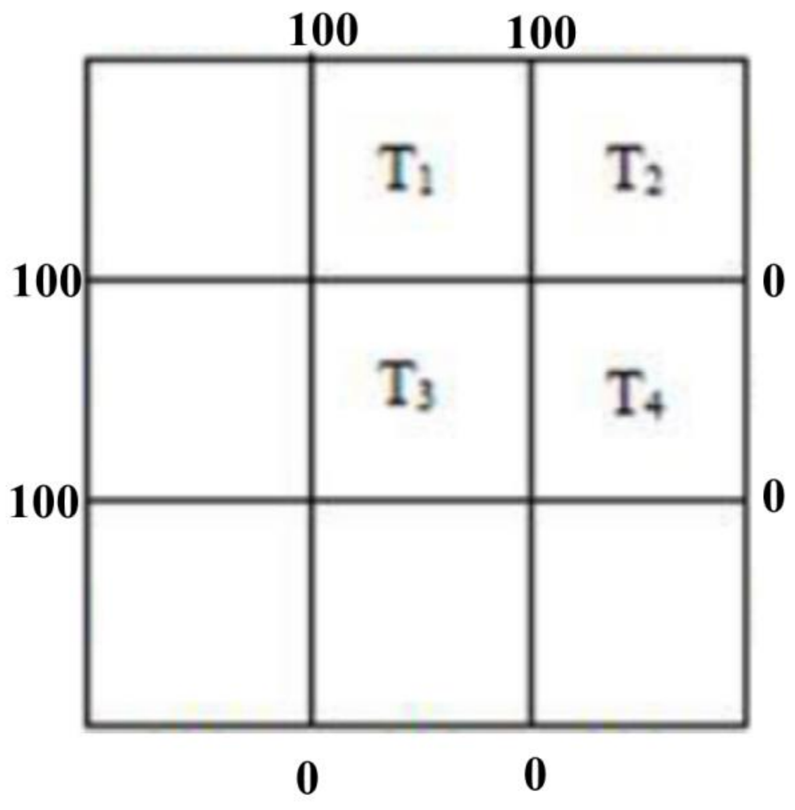
$f_1 = -\frac{11}{2}$

Using these values in eq. (ii)

$f_4 = -\frac{11}{2}$

$\therefore\ f_1 = -\frac{11}{2}, f_2 = -\frac{43}{4}, f_3 = -\frac{13}{4}$ & $f_4 = -\frac{11}{2}$

2.** The steady state two dimensional heat-flow in a metal plate is defined by* $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$ ***Given the boundary conditions as sown in figure below, find the temperatures at interior points T_1, T_2, T_3 ***and*** T_4.



Using the difference equation for the Laplace equation,

At point T_1

$$T_1 = \frac{1}{4}(100 + 100 + T_2 + T_3)$$

Or, $4T_1 - T_2 - T_3 = 200$ .... (i)

At point T_2

$$T_2 = \frac{1}{4}(100 + 0 + T_1 + T_4)$$

Or, $4T_2 - T_1 - T_4 = 100$ .... (ii)

At point T_3

$$T_3 = \frac{1}{4}(100 + 0 + T_1 + T_4)$$

Or, $4T_3 - T_1 - T_4 = 100$ ……………. (iii)

<u>At point T_4</u>

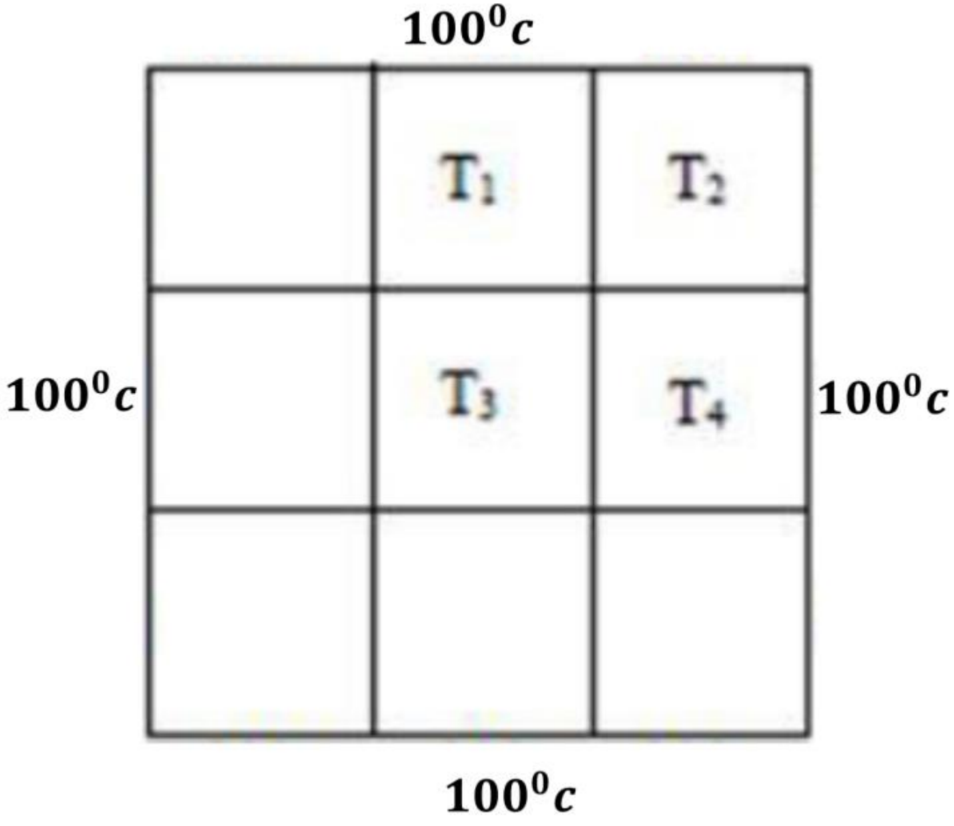
$$T_4 = \frac{1}{4}(0 + 0 + T_2 + T_3)$$

Or, $4T_4 - T_2 - T_3 = 0$ ……………. (iv)

Solving equation (i), (ii), (iii) & (iv) we get;

$$T_1 = 75$$
$$T_2 = 50$$
$$T_3 = 50$$
$$T_4 = 25$$

3. ***The steady-state two dimensional heat flow in a metal plate is defined by*** $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. ***Consider a metal plate of size*** $30cm \times 30cm$***, the boundaries of which are held at*** $100^0 c$. ***Calculate the temperature at interior points of the plate. Assume the grid size of*** $10cm \times 10cm$.



Solution:

Let T_1, T_2, T_3 & T_4 are the values of interior grid point.

Using the difference equation for the Laplace equation,

At point T_1

$$T_1 = \frac{1}{4}(100 + 100 + T_2 + T_3)$$

Or, $4T_1 - T_2 - T_3 = 200$ ……………. (i)

At point T_2

$$T_2 = \frac{1}{4}(100 + 100 + T_1 + T_4)$$

Or, $4T_2 - T_1 - T_4 = 200$ ……………. (ii)

At point T_3

$$T_3 = \frac{1}{4}(100 + 100 + T_1 + T_4)$$

Or, $4T_3 - T_1 - T_4 = 200$ ……………. (iii)

At point T_4

$$T_4 = \frac{1}{4}(100 + 100 + T_2 + T_3)$$

Or, $4T_4 - T_2 - T_3 = 200$ ……………. (iv)

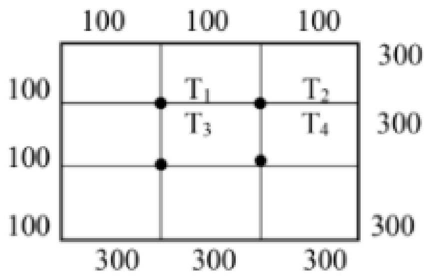
Solving equation (i), (ii), (iii) & (iv) we get;

$$T_1 = 100$$
$$T_2 = 100$$
$$T_3 = 100$$
$$T_4 = 100$$

6. Derive a difference equation to represent a Poison's equation. Solve the Poison's equation $\nabla^2 f = 2x^2y^2$ over the domain $0 \le x \le 3, 0 \le y \le 3$ with f = 0 on the boundary and h = 1.(3 + 5) asked in 2066

6. a) How can you obtain numerical solution of a partial differential equation? Explain.(3) asked in 2069

b) The steady-state two-dimensional heat-flow in a metal plate is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$.Given the boundary conditions as shown in figure below, find the temperature at interior points T_1, T_2, T_3 and T_4. (5)



6. Solve the equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 3x^2y$ over the square domain $0 \le x \le 1.5$ and $0 \le y \le 1.5$ with f=0 on the boundary [Take h = 0.5]. (8) asked in 2070

6. Write the finite difference formula for solving Poisson's equation. Hence solve the Poisson's equation $\nabla^2 f = 2x^2y^2$ over the domain $0 \le x \le 3$ and $0 \le y \le 3$ with f = 0 on the boundary and h = 1. (1 + 7) asked in 2071

6. What do you understand by the partial differential equation? Illustrate it with practical example and derive difference equation. (8) asked in 2072

OR

Find the solution of following differential equations using Taylor series method.

$y' = (x^3 + xy^2)e^{(-x)}$, $y(0) = 1$, to find y at x = 0.1, 0.2, 0.3.

6. What do you understand by the partial differential equation? Illustrate it with practical example and derive difference equation. (8) asked in 2072

OR

Find the solution of following differential equations using Taylor series method.

$y' = (x^3 + xy^2)e^{(-x)}$, $y(0) = 1$, to find y at x = 0.1, 0.2, 0.3.

6. Write the finite difference formula for solving Poisson's equation. Hence solve the Poisson equation $\nabla^2 f = 3x^2 y$ over the domain $0 \le x \le 1.5$ and $0 \le y \le 3$ with f = 0 on the boundary and h = 0.5 (1 + 7) asked in 2073

6. Define partial differential equation. Discuss Laplace's equation along with its derivation.(2 + 6) asked in 2074

6. (a) Derive a difference equation to solve Laplace equation.(3) asked in 2075(Old Course)

(b) The steady-state two-dimensional heat-flow in a metal plate is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. Given the boundary conditions as shown in figure below. find the temperatures at interior points A, B, C, and D.(5)